On the Evolution of Minimal-Volume, Sufficient-Probability Sets for Stochastic Paths

Ryan T. White

Florida Institute of Technology

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University of Oxford

An $\mathbb{R}^d$-valued Random Variable

- Setting: a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Consider a random variable $X : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B})$
  - $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^d$

Minimal-Volume, Sufficient-Probability Sets

- Fix $\alpha \in [0, 1]$.
- Sets containing X with sufficient probability

$$\mathscr{S} = \{S \in \mathcal{B} : \mathbb{P}(X \in S) \geq \alpha\}$$

- Minimum-volume sufficient-probability sets (or α-MVSPs)

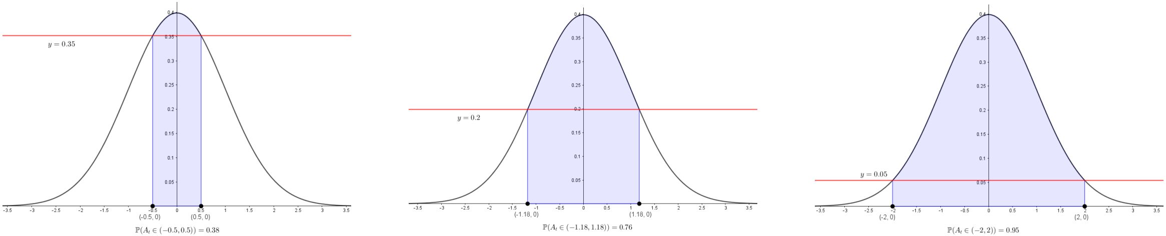
$$\mathscr{M} = \{\arg\min\{|M| : M \in \mathscr{S}\}\}$$

where $|\cdot|$ is the Lebesgue measure.

Superlevel Sets

- The superlevel sets of the pdf f of X are

$$L^y = \{x \in \mathbb{R} : f(x) \geq y\}$$



$L^{0.35} = (-0.5, 0.5)$ $\quad$ $L^{0.2} = (-1.18, 1.18)$ $\quad$ $L^{0.05} = (-2, 2)$

Figure: Superlevel sets of pdf f with $\mathbb{P}(X \in L^y) = \int_{L^y} f(x)\,dx$ in blue

Probability Set-Density

- Probability set-density

$$D : \{B \in \mathcal{B} : |B| > 0\} \to [0, \sup f]$$

where

$$D(B) = \frac{\mathbb{P}(X \in B)}{|B|}$$

- $D(B)$ is the probability $X \in B$ (mass) divided by the Lebesgue measure of the region B (volume)

Minimum Volume Class of Sets (MVCS)

- MVCS of X for $\alpha \in [0,1]$ are

$$MVC : [0,1] \to \mathscr{P}(\mathcal{B})$$

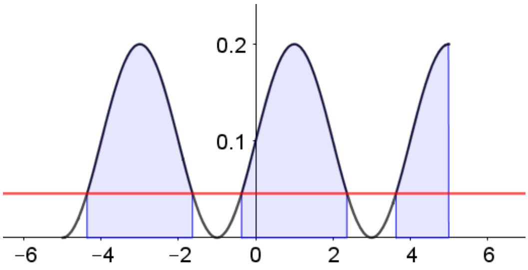
where

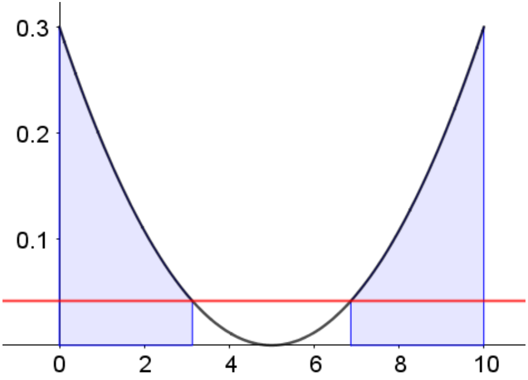
$$MVC(\alpha) = \{\arg\max\{D(B) : \mathbb{P}(X \in B) = \alpha\}\}$$

- Requires $= \alpha$ rather than $\geq \alpha$, so certain pdfs fail to have MVCS
  - We assume f is continuous with $f' = 0$ on null sets of the domain.

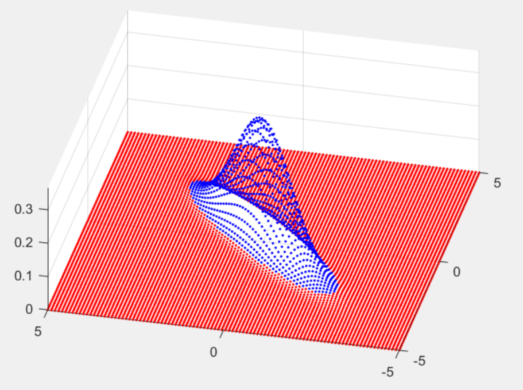
α-MVCs and Superlevel Sets

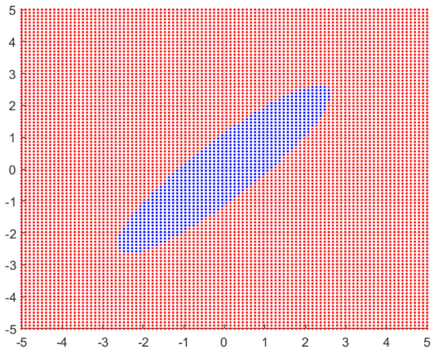
- For these examples, 0.95-MV coincide with superlevel sets



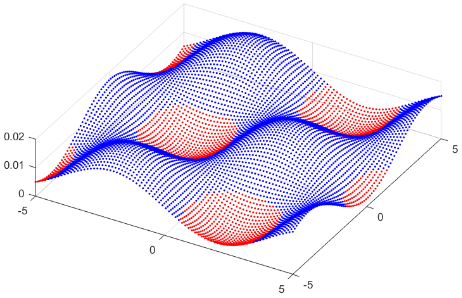


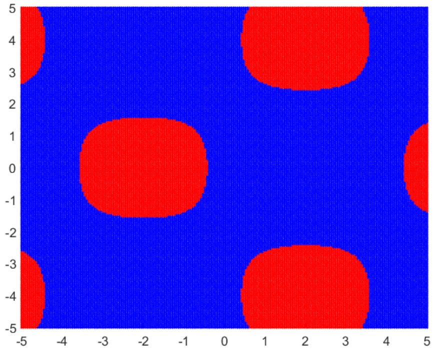
Example: Bivariate Normal





- $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$, and $\rho = 0.9$
- Code: sort pdf on a mesh and add tiny regions to build a 0.95-MVSP set





- Bivariate pdf $f(x_1, x_2) = \frac{\sin(n\pi x_1)\cos(n\pi x_2)+1}{4L^2}$ on $[-L, L] \times [-L, L]$ for $L = 5$, $n = \frac{1}{4}$

α-MV-Superlevel Set Equivalence

Theorem (Garcia et al. [2003])

A superlevel set L^y of f has probability $\alpha \in (0,1]$, $\mathbb{P}(X \in L^y) = \alpha$, if and only if the superlevel set is a MVCS of X, $L^y \in MVC(\alpha)$.

- Probability mass functions work too
- Some similar work exists from [Polonik, 1995]
- Prior results for continuous f exist from outside probability literature [Nguyen and Kreinovich, 1999]

Toward Stochastic Processes

- Garcia et al. [2003] focused on a **random variable**
- Can we exploit the MVC-superlevel set equivalence a **stochastic process**?
  - Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.
  - Let $(A_t, t \geq 0) : (\Omega, \mathcal{F}_t) \to (\mathbb{R}^d, \mathcal{B})$ be a stochastic process

Example: Brownian Motion

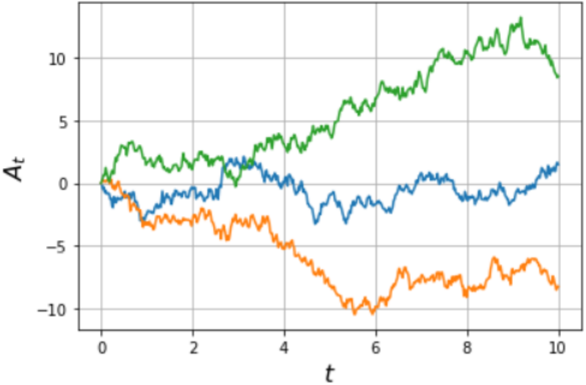


Figure: A_t is Brownian motion valued in $\mathbb{R}^1$

Example: Brownian Motion

- $A_t \sim \mathcal{N}(0, t)$, $\alpha = 0.95$

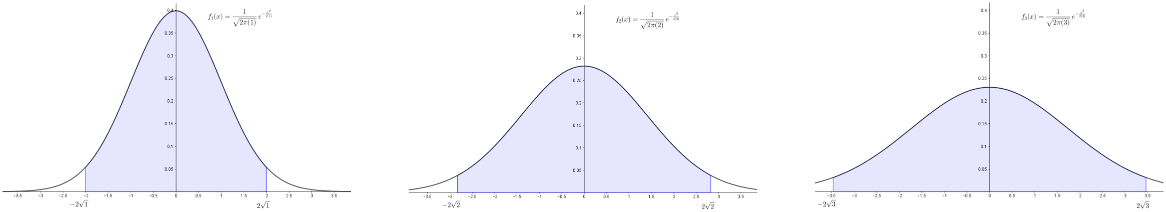
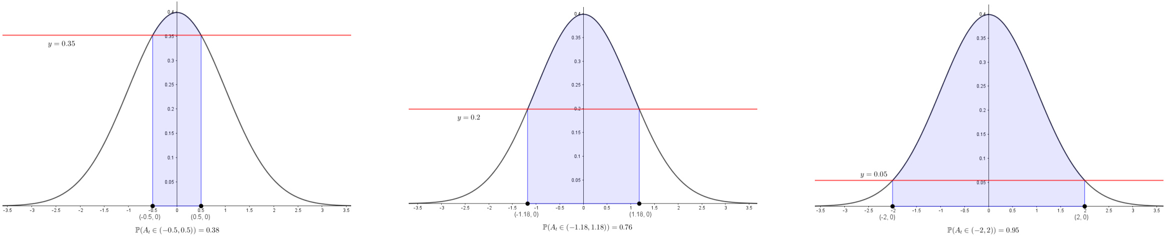


Figure: Probability density functions (pdfs) f_t of A_t with 0.95-MVSPs and probabilities

- For fixed t, a 0.95-MVSP set is $(-2\sqrt{t}, 2\sqrt{t})$.

Example: Brownian Motion

- If we take a horizontal line at $y = \sup f_t$ and move it down and integrate, eventually we get $\alpha = 0.95$.



$L_1^{0.35} = (-0.5, 0.5)$ $L_1^{0.2} = (-1.18, 1.18)$ $L_1^{0.05} = (-2, 2)$

Figure: Superlevel sets of pdf f_1 with $\mathbb{P}(A_1 \in L_1^y) = \int_{L_1^y} f_1(x)\,dx$ in blue

MVSP-Superlevel Set Equivalence for Brownian Motion

- For Brownian motion,

$$\mathbb{P}\left(A_t \in L_t^{f_t(2\sqrt{t})}\right) = \mathbb{P}\left(A_t \in \left(-2\sqrt{t}, 2\sqrt{t}\right)\right) = 0.95$$

- **0.95-MVSPs are superlevel sets of f_t for Brownian motion**

Example: Brownian Motion

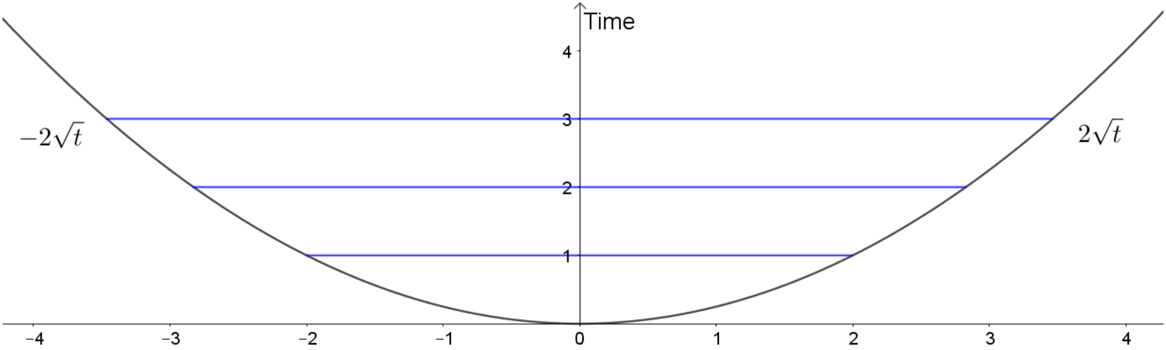
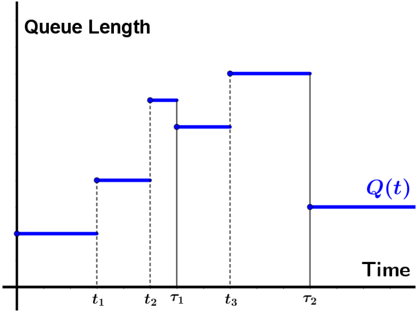


Figure: The evolution of 0.95-MVSPs ($f_t(2\sqrt{t})$-superlevel sets) for Brownian motion

A Queueing Process

- Q_t on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$
- Entering customers: batches X arrive at $t_1 < t_2 < \ldots$
- Exiting customers: batches Y depart at $\tau_1 < \tau_2 < \ldots$



QUEUE TYPE: $M^X/G^Y/1$ with several operational policies

Two Common Problems with Queueing Systems

- Switchovers can be costly

- Resources are wasted when the system is off

GOAL: Minimize switchovers and do secondary tasks when possible while serving customers efficiently

N-Policy: Reducing Switchovers

POLICY:

1. If $Q_t < N$ and the system is off, wait until it reaches N.
2. Else, serve customers.

- Classical switchover mitigation technique [Yadin and Naor, 1963]
  - Q_t small $\implies$ queue is likely exhausted quickly and system turns off
  - Q_t large $\implies$ queue will persist, system works continuously

BIG CON: customers must wait sometimes

r-R-Quorum: Completing Secondary Tasks

POLICY:

1. If $Q_t = 0$, batch secondary service
2. If $0 < Q_t < r$, parallel service – single primary service, batch secondary service
3. If $Q_t > r$, primary service to batch of $\min\{\text{queue}, R\}$

- Related to some classical queueing ideas
  - r-quorum [Neuts, 1967]
  - Hysteretic control [Loris-Tegham, 1978]

MAIN BENEFITS: less primary waiting , secondary work done

Hysteretic Control

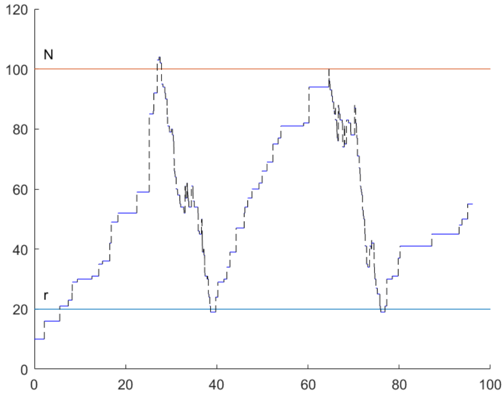
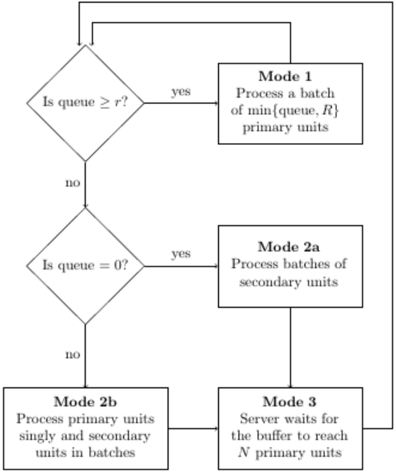


Figure: A path of Q_t with r and N control levels

Queueing System Structure



Prior Results on $Q(\tau_\rho)$

- [Dshalalow et al., 2019]: functionals of changes during each mode

$$\Phi(u,v,w,\theta)=\mathbb{E}\left[u^{\text{primary served}}v^{\text{secondary served}}w^{\text{primary arrivals}}e^{-\theta(\text{duration})}\right]$$

with modified z-transforms

$$\Phi \xrightarrow{\text{Transform}} \Psi \xrightarrow{\text{Assumptions on the system}} \Psi \text{ (convenient form)} \xrightarrow{\text{Inverse}} \Phi \text{ (tractable)}$$

- Moments and marginal distributions
- Transition probability matrix
- Ergodicity conditions, similar to [Abolnikov and Dukhovny, 1991]
- Stationary distribution
- Mean stationary service cycle

Prior Results on Q_t

- Let $Q_t = (A_t, B_t) =$ (primary queue length in mode, secondary units processed in mode)
- For each mode with (random) duration τ_ρ, we found [White and Dshalalow, 2019]

$$\Phi(s,u,v,\theta) = \int_{t\geq 0} e^{-st}\mathbb{E}\left[z^{A_t}\xi^{B_t}e^{-\theta\tau_\rho}\mathbb{1}_{[0,\tau_\rho)}(t)\right]dt$$

- It is simple to find

$$\Phi \xrightarrow{\text{inverse}} \mathbb{E}\left[u^{A_t}v^{B_t}e^{-\theta\tau_\rho}\mathbb{1}_{[0,\tau_\rho)}(t)\right] \xrightarrow{\text{dist. assumptions}}$$ **time-dependent** moments, marginal/joint distributions

What's Missing?

- Distributions independent of mode

$$\mathbb{E}\left[u^{A_t} v^{B_t} e^{-\theta \tau_\rho} \mathbb{1}_{[0,\tau_\rho)}(t)\right]$$

- No clear path analytically $\rightarrow$ we CAN simulate
- Can we optimize parameters for efficiency?
- Where does the system get "stuck"?
- What's the distribution on service exit?

Drop Process ξ_t

- Let $\xi_t = A_t$ upon each primary service exit and otherwise constant

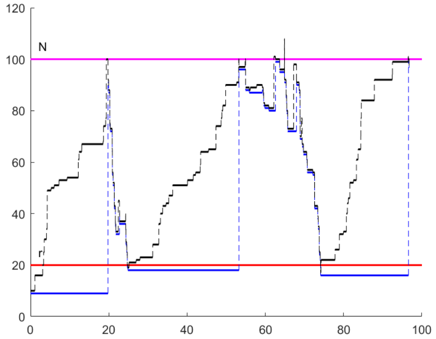
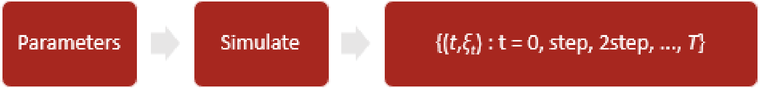


Figure: Black: Q_t, Blue: ξ_t for rN-policy without Mode 2

Simulating the Queue



- 10,000 sims with $T = 50$, we get many sets $\{(t, \xi_t) : t = 0, \text{step}, 2\text{step}..., 50\}$

  $\rightarrow$ empirical distribution at each t

Empirical Densities of ξ_t

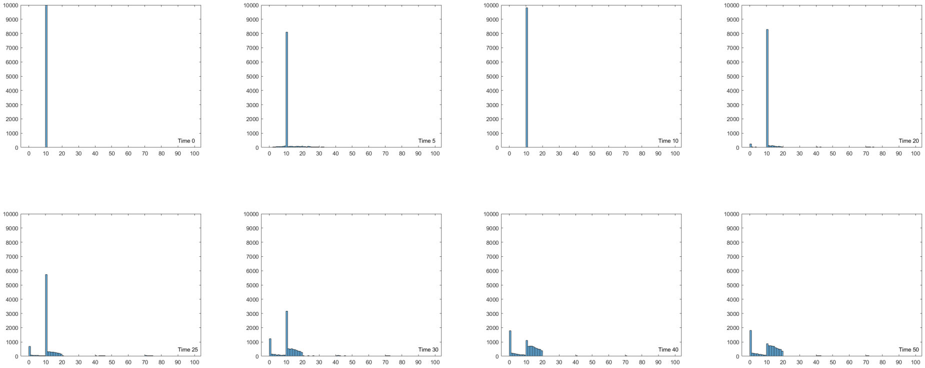
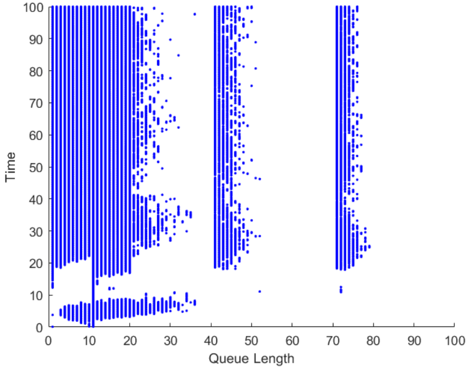
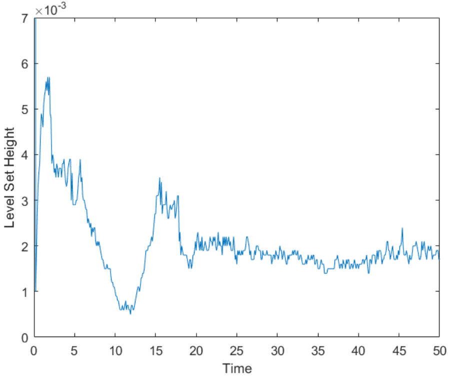


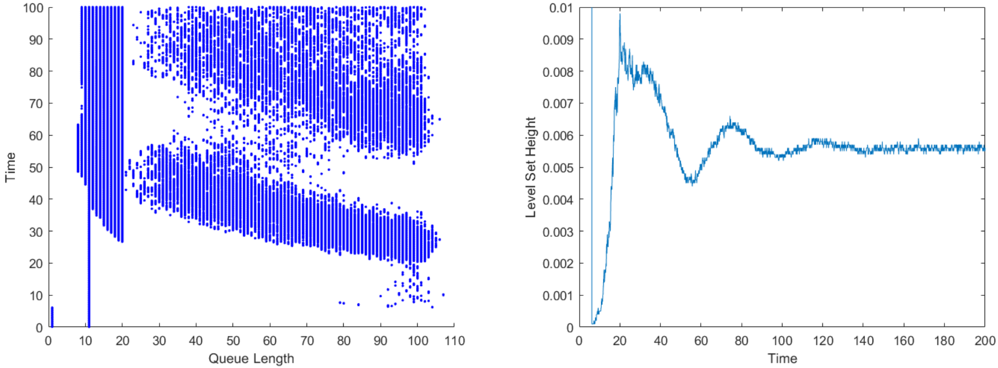
Figure: The densities of ξ_t wit $r = 20,\ N = 100,\ R = 30,\ S = 20,\ Q_0 = 10$

0.95-MVSP-Superlevel Set Evolution





0.70-MVSP-Superlevel Set Evolution



- From a simpler model [Al-Obaidi and Dshalalow, 2020]
- No Mode 2, r and N control levels, primary batches geometric.

Example: Multidimensional Brownian Motion

- Let B_t be standard Brownian motion
- B_t is multivariate normal with mean $\mathbf{0}$ and covariance $\Sigma = diag(t, ..., t)$

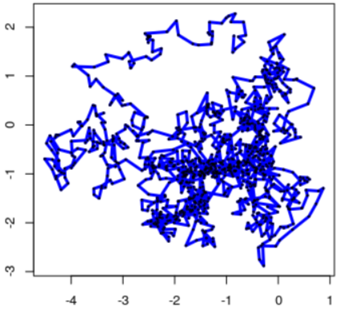
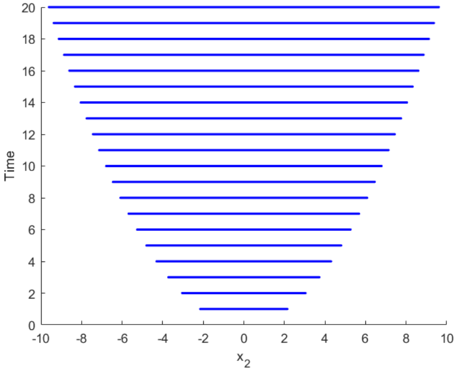
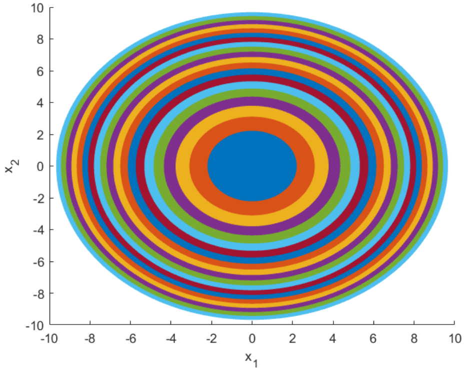


Figure: A simulated path of 2D Brownian Motion

- We can search for the α-level sets and α-MVSPs with code by sorting an empirical pdf on a 2D mesh



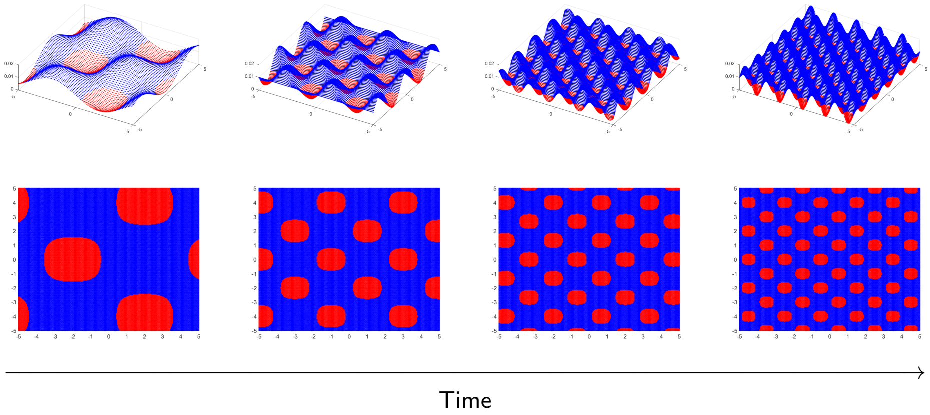
(a) Level sets with $\mathbb{P}\left(B_t \in L_t^y\right) = 0.95$



(b) Plots of MVSPs expand as time progresses

New Work

- Let $n(t) = t + \frac{1}{4}$ for $0 \leq t \leq \frac{3}{4}$ in $f(x_1, x_2) = \frac{\sin(n\pi x_1)\cos(n\pi x_2)+1}{4L^2}$, $L = 5$



- Use **probability flux** across the MVSP boundaries to reduce computational stress → more dimensions

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