Random Walks on Random Lattices

An Operational Calculus Approach

Ryan White

MTH/ORP Graduate Student Seminar

September 30, 2014

Probability Space

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space if

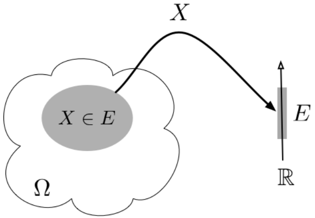
- Ω is any set (the sample space)
- $\mathcal{F}$ is a σ-algebra (a collection of subsets of Ω representing events)
- $\mathbb{P}$ is a probability measure
    - $\mathbb{P} : \mathcal{F} \to [0, 1]$ assigns a probability to each event

In short, a probability space is just a measure space where $\mathbb{P}(\Omega) = 1$

Random Variables

A real-valued RV is a **measurable** function $X : (\Omega, \mathcal{F}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$

- Technically, for each $E \in \mathcal{B}(\overline{\mathbb{R}})$, $X^{-1}(E) = \{X \in E\} \in \mathcal{F}$



- E.g., if $E = [-\infty, t]$, the CDF of X is

$$\mathbb{P}\{X \in E\} = \mathbb{P}\{X \in [-\infty, t]\} = \mathbb{P}(X \le t)$$

Expectation

- If X is $\mathbb{P}$-integrable, $X \in L^1(\mathbb{P})$, its expectation is defined as

$$\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}$$

- This generalizes the possibly more familiar notions of expectation

$$\mathbb{E}[X] = \sum_{j \in \mathbb{Z}} x^j \mathbb{P}\{X = j\} \qquad \text{(for discrete RVs)}$$

$$\mathbb{E}[X] = \int_\mathbb{R} x f(x) \, dx \qquad \text{(for continuous RVs)}$$

Properties of RVs and Expectation

For $g, h \in C^{-1}\left(\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}})\right)$ (measurable functions), X and Y RVs

1. $g \circ X$ is a RV
2. $\mathbb{E}[\mathbf{1}_A(\omega)] = \int_A d\mathbb{P} = \mathbb{P}\{A\}$
3. X and Y independent $\implies \mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$

Conditional Expectation

The conditional expectation $\mathbb{E}[X|Y]$ is the expectation of X given Y

1. $\mathbb{E}[\mathbb{E}[g(X)|Y]] = \mathbb{E}[g(X)]$ (Double Expectation)

2. $\mathbb{E}[g(X)f(X,Y)|X] = g(X)\mathbb{E}[f(X,Y)|X], a.s.$

Probability Transforms

- The probability-generating function (PGF) of a discrete RV X is $g(z) = \mathbb{E}[z^X]$, $\|z\| \leq 1$
  - $\mathbb{P}\{X = k\} = \frac{g^{(k)}(0)}{k!}$ (Distribution)
- The Laplace-Stieltjes transform (LST) of a continuous nonnegative RV X is $L(\theta) = \mathbb{E}[e^{-\theta X}]$, $Re(\theta) \geq 0$
  - $\mathbb{P}\{X \leq t\} = \mathcal{L}_\theta^{-1}\left\{\frac{L(\theta)}{\theta}\right\}(t)$ (CDF)
  - $\mathbb{E}[X^k] = (-1)^k \lim_{\theta \to 0} L^{(k)}(\theta)$ (Moments)

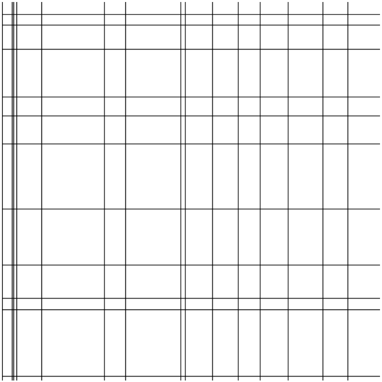
Papers

We follow the ideas of the papers

- J. H. Dshalalow and R. White. On Reliability of Stochastic Networks. *Neural, Parallel, and Scientific Computations*, 21 (2013) 141-160
- J. H. Dshalalow and R. White. On Strategic Defense in Stochastic Networks. *Stochastic Analysis and Applications*, accepted for publication (2014)

Random Walks on Random Lattices

(a) Random Walk



(b) Random Lattice

Stochastic Network Cumulative Loss Model

1. $\{t_1, t_2, ...\}$ – point process of attack times
2. n_k – *iid* number of nodes lost at t_k with PGF $g(z) = E\left[z^{n_1}\right]$.
3. w_{jk} – *iid* weight per node with LST $l(u) = E\left[e^{-uw_{11}}\right]$
   - $w_k = \sum_{j=1}^{n_k} w_{jk}$ – total weight lost at t_k

The Cumulative Loss Process

We use a (multidimensional) compound Poisson random measure of rate λ, i.e., for $E \in \mathcal{B}(\mathbb{R}_{\geq 0})$,

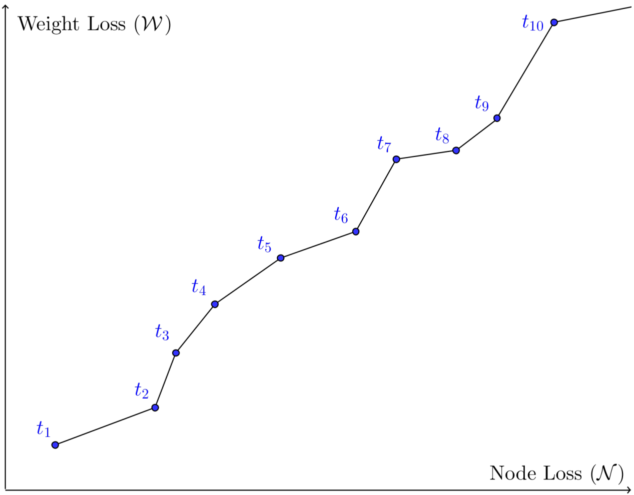
- $\eta(E) = \sum_{k=1}^{\infty} (n_k, w_k)\varepsilon_{t_k}(E)$
- ε_{t_k} is the Dirac (point mass) measure,

$$\varepsilon_{t_k}(E) = \begin{cases} 1 & : t_k \in E \\ 0 & : t_k \notin E \end{cases}$$

- $\eta([0,t])$ is the cumulative losses of nodes and weight up to time t

The Cumulative Loss Process (cont.)

$\eta([0,t])$ is a monotone increasing process on $\mathbb{N} \times \mathbb{R}_+$,



Delayed Observation

The process is observed upon a delayed renewal process

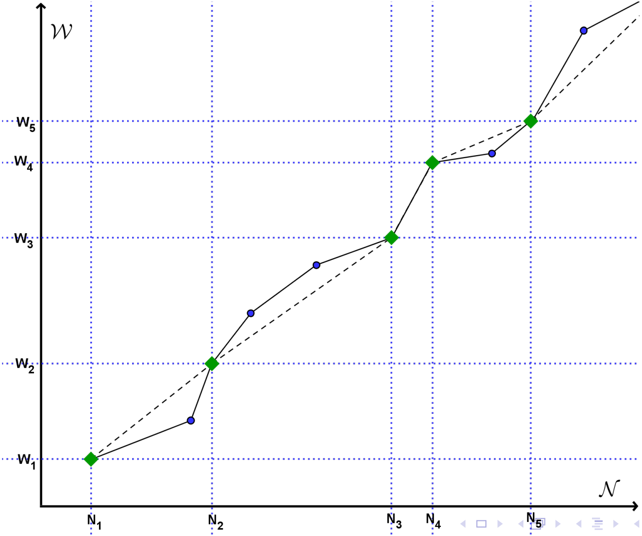
- $\mathcal{T} = \sum_{n=0}^{\infty} \varepsilon_{\tau_n}$
  - $\Delta_k = \tau_k - \tau_{k-1}$ are *iid* with LST $L(\theta)$, $k \in \mathbb{N}$
  - $\tau_0 = \Delta_0$ is independent of Δ_k, $k \geq 1$

The process of interest is the **embedded process**

- $Z = \sum_{n=0}^{\infty} \eta((\tau_{n-1}, \tau_n])\varepsilon_{\tau_n} = \sum_{n=0}^{\infty} (X_n, Y_n)\varepsilon_{\tau_n}$
  - $(N_n, W_n) = Z([0, \tau_n])$, the value of the process at τ_n

Delayed Observation as an Embedded Process

We consider the (green) embedded process



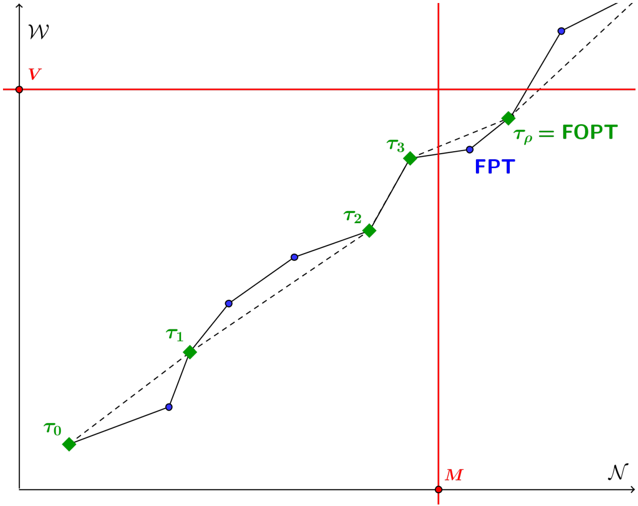
Observed Threshold Crossings

- Each component has a threshold: M for nodes, V for weights
- The **first observed passage time** (FOPT) is τ_ρ, where

$$\rho = \min\{n : N_n > M \text{ or } W_n > V\}$$

- Threshold crossing by the observed (embedded) process Z

Observed Threshold Crossings (FPT vs FOPT)



Joint Transform of Each Observation Epoch

- $\gamma(z, v, \theta) = \mathbb{E}\left[z^{X_1} e^{-vY_1} e^{-\theta\Delta_1}\right] = \cdots$

$$\cdots = L\left[\theta + \lambda - \lambda g\left(zl\left(v\right)\right)\right]$$

- For Δ_0, the result is $\gamma_0(z, v, \theta) = L_0[\theta + \lambda - \lambda g(zl(v))]$

Joint Transform of Each Observation Epoch

- $$\begin{aligned}\gamma(z,v,\theta) &= \mathbb{E}\left[z^{X_1}e^{-vY_1}e^{-\theta\Delta_1}\right]\\ &= \mathbb{E}\left[e^{-\theta\Delta_1}\mathbb{E}\left[z^{X_1}e^{-vY_1}\middle|\Delta_1\right]\right] && \text{(Double Expectation)}\\ &= \mathbb{E}\left[e^{-\theta\Delta_1}\mathbb{E}\left[z^{n_1}e^{-v\left(w_{11}+\ldots+w_{n_1 1}\right)}\times\cdots\times z^{n_J}e^{-v\left(w_{1J}+\ldots+w_{n_J J}\right)}\middle|\Delta_1\right]\right]\\ &= \mathbb{E}\left[e^{-\theta\Delta_1}\mathbb{E}\left[\left(g\left(zl\left(v\right)\right)\right)^J\middle|\Delta_1\right]\right] && (n_k\text{'s and } w_{jk}\text{'s are } iid)\\ &= \mathbb{E}\left[e^{-(\theta+\lambda-\lambda g(zl(v)))\Delta_1}\right] && (J \text{ is Poisson with parameter } \lambda\Delta_1)\\ &= L\left[\theta+\lambda-\lambda g\left(zl\left(v\right)\right)\right]\end{aligned}$$
- For Δ_0, the result is $\gamma_0(z,v,\theta) = L_0[\theta+\lambda-\lambda g(zl(v))]$

Functional of Interest

- We seek a joint functional of the of the process upon $\tau_{\rho-1}$ and τ_ρ,

$$\Phi(\alpha_0, \alpha, \beta_0, \beta, h_0, h) = \mathbb{E}\left[{\alpha_0}^{N_{\rho-1}} \alpha^{N_\rho} e^{-\beta_0 W_{\rho-1} - \beta W_\rho} e^{-h_0 \tau_{\rho-1} - h\tau_\rho}\right]$$

Whi Φ?

$$\Phi(\alpha_0, \alpha, \beta_0, \beta, h_0, h) = \mathbb{E}\left[{\alpha_0}^{N_{\rho-1}} \alpha^{N_\rho} e^{-\beta_0 W_{\rho-1} - \beta W_\rho} e^{-h_0 \tau_{\rho-1} - h\tau_\rho}\right]$$

Φ leads to marginal PGFs/LSTs,

- $\Phi(1, \alpha, 0, 0, 0, 0) = \mathbb{E}\left[\alpha^{N_\rho}\right]$
- $\Phi(1, 1, \beta_0, 0, 0, 0) = \mathbb{E}\left[e^{-\beta_0 W_{\rho-1}}\right]$
- $\Phi(1, 1, 0, 0, 0, h) = \mathbb{E}\left[e^{-h\tau_\rho}\right]$

which lead to moments and distributions of components of the process upon $\tau_{\rho-1}$ and τ_ρ

Goal and Operational Calculus Strategy

- The goal is to derive Φ in an analytically or numerically tractable form
- Strategy to derive Φ,

$$\Phi \xrightarrow{\mathcal{H}} \Psi \xrightarrow{\text{Assumptions}} \Psi \text{ (convenient form)} \xrightarrow{\mathcal{H}^{-1}} \Phi \text{ (tractable)}$$

for an operator $\mathcal{H}$ to be introduced next

$\mathcal{H}$ Operator

We introduce an operator,

$$\mathcal{H}_{pq} = \mathcal{LC}_q \circ D_p$$

where

$$\mathcal{LC}_q(g(q))(y) = y\mathcal{L}_q(g(q)) = y \int_{q=0}^{\infty} g(q)e^{-qy}\, dq$$

$$D_p\{f(p)\}(x) = (1-x)\sum_{p=0}^{\infty} x^p f(p), \|x\| < 1$$

for a function $g(q)$ and a sequence $f(p)$.

Inverse Operator

- D_p has an inverse which can restore f,
  - $\mathcal{D}_x^k\left(\cdot\right) = \lim_{x\to 0} \frac{1}{k!}\frac{\partial^k}{\partial x^k}\left(\frac{1}{1-x}(\cdot)\right),\ k \in \mathbb{N}$
  - $\mathcal{D}_x^k\left(D_p\{f(p)\}(x)\right) = f(k)$
- A key asset of the inverse:

$$\mathcal{D}_x^k\left(\sum_{j=0}^{\infty} a_j x^j\right) = \sum_{j=0}^{k} a_j$$

- The inverse of the composition $\mathcal{H}_{pq}$ is

$$\mathcal{H}_{xy}^{-1}\left(\Psi(x,y)\right)(p,q) = \mathcal{L}_y^{-1}\left(\frac{1}{y}\mathcal{D}_x^p\left(\Psi(x,y)\right)\right)(q)$$

Decomposition of Φ

- Let $\mu = \inf\{n : N_n > M\}$, $\nu = \inf\{n : W_n > V\}$
- Decompose Φ as

$$\Phi = \mathbb{E}\left[{\alpha_0}^{N_{\rho-1}} \alpha^{N_\rho} e^{-\beta_0 W_{\rho-1} - \beta W_\rho} e^{-h_0 \tau_{\rho-1} - h\tau_\rho}\right]$$

$$= \mathbb{E}\left[\alpha_0^{N_{\mu-1}} \alpha^{N_\mu} e^{-\beta_0 W_{\mu-1} - \beta W_\mu - h_0 \tau_{\mu-1} - h\tau_\mu} \mathbf{1}_{\{\mu<\nu\}}\right]$$

$$+ \mathbb{E}\left[\alpha_0^{N_{\mu-1}} \alpha^{N_\mu} e^{-\beta_0 W_{\mu-1} - \beta W_\mu - h_0 \tau_{\mu-1} - h\tau_\mu} \mathbf{1}_{\{\mu=\nu\}}\right]$$

$$+ \mathbb{E}\left[\alpha_0^{N_{\nu-1}} \alpha^{N_\nu} e^{-\beta_0 W_{\nu-1} - \beta W_\nu - h_0 \tau_{\nu-1} - h\tau_\nu} \mathbf{1}_{\{\mu>\nu\}}\right]$$

$$= \Phi_{\mu<\nu} + \Phi_{\mu=\nu} + \Phi_{\mu>\nu}$$

- We will derive $\Phi_{\mu<\nu}$ of the **confined process** on the trace σ-algebra $\mathcal{F} \cap \{\mu < \nu\}$

Decomposing Further, Applying $\mathcal{H}_{pq}$

- Introduce $\mu(p) = \inf\{j : N_j > p\}$, $\nu(q) = \inf\{k : W_k > q\}$

$$\Phi_{\mu(p)<\nu(q)} = \sum_{j\geq 0} \sum_{k>j} \mathbb{E}\left[\alpha_0^{N_{j-1}} \alpha^{N_j} e^{-\beta_0 W_{j-1} - \beta W_j - h_0 \tau_{j-1} - h\tau_j} \mathbf{1}_{\{\mu(p)=j,\, \nu(q)=k\}}\right]$$

- Next, apply $\mathcal{H}_{pq}$ to $\Phi_{\mu(p)<\nu(q)}$

- By Fubini's Theorem, $\mathcal{H}_{pq}$ can be interchanged with the two series and expectation (an integral w.r.t. $\mathbb{P}$)

Calculating $\mathcal{H}_{pq}\left(\mathbf{1}_{\{\mu(p)=j,\nu(q)=k\}}\right)(x,y)$

$$\mathcal{H}_{pq}\left(\mathbf{1}_{\{\mu(p)=j,\nu(q)=k\}}\right)(x,y)$$

$$= \mathcal{H}_{pq}\left(\mathbf{1}_{\{N_{j-1}\le p\}}\mathbf{1}_{\{N_j>p\}}\mathbf{1}_{\{W_{k-1}\le q\}}\mathbf{1}_{\{W_k>q\}}\right)(x,y)$$

$$= y(1-x)\sum_{p=N_{j-1}}^{N_j-1} x^p \int_{q=W_{k-1}}^{W_k} e^{-yq}dq$$

$$= \left(x^{N_{j-1}} - x^{N_j}\right)\left(e^{-yW_{k-1}} - e^{-yW_k}\right)$$ (Partial geometric series)

Independent Increments in $\mathcal{H}_{pq}\left(\Phi_{\mu(p)<\nu(q)}\right)(x,y)$

$$\mathcal{H}_{pq}\left(\Phi_{\mu(p)<\nu(q)}\right)(x,y) = \sum_{j\geq 0}\sum_{k>j}\mathbb{E}\big[\alpha_0^{N_{j-1}}\alpha^{N_j}e^{-\beta_0 W_{j-1}-\beta W_j-h_0\tau_{j-1}-h\tau_j}$$

$$\times\left(x^{N_{j-1}}-x^{N_j}\right)\left(e^{-yW_{k-1}}-e^{-WB_k}\right)\big]$$

$$= \sum_{j\geq 0}\mathbb{E}\left[(\alpha_0\alpha x)^{N_{j-1}}e^{-(\beta_0+\beta+y)W_{j-1}-(h_0+h)\tau_{j-1}}\right]$$

$$\times\mathbb{E}\left[\alpha^{X_j}\left(1-x^{X_j}\right)e^{-(\beta+y)Y_j-h\Delta_j}\right]$$

$$\times\sum_{k>j}\mathbb{E}\left[e^{-y(Y_{j+1}+\ldots+Y_{k-1})}\right]\mathbb{E}\left[1-e^{-yY_k}\right]$$

Simplifying $\mathcal{H}_{pq}\left(\Phi_{\mu(p)<\nu(q)}\right)(x,y)$

- Denote

$$\begin{aligned}\gamma &= \gamma\left(\alpha_0\alpha x, \beta_0+\beta+y, h_0+h\right)\\ \Gamma &= \gamma\left(\alpha x, \beta+y, h\right)\\ \Gamma^1 &= \gamma\left(\alpha, \beta+y, h\right)\end{aligned}$$

- Then we can simplify the expectations as

$$\mathbb{E}\left[(\alpha_0\alpha x)^{N_{j-1}}\, e^{-(\beta_0+\beta+y)W_{j-1}-(h_0+h)\tau_{j-1}}\right] = \begin{cases} 1, & j=0\\ \gamma_0\gamma^{j-1}, & j>0\end{cases}$$

$$\mathbb{E}\left[\alpha^{X_j}\left(1-x^{X_j}\right)e^{-(\beta+y)Y_j-h\Delta_j}\right] = \begin{cases} \Gamma_0^1-\Gamma_0, & j=0\\ \Gamma^1-\Gamma, & j>0\end{cases}$$

$$\mathbb{E}\left[e^{-y(Y_{j+1}+\ldots+Y_{k-1})}\right] = \gamma^{k-1-j}\left(1,y,0\right),\ k>j\geq 0$$

$$\mathbb{E}\left[1-e^{-yY_k}\right] = 1-\gamma\left(1,y,0\right),\ k>j\geq 0$$

Deriving $\Phi_{\mu<\nu}$

- Implementing the notation, we see the j and k series become

$$\mathit{\Gamma}_0^1 - \mathit{\Gamma}_0 + (\mathit{\Gamma}^1 - \mathit{\Gamma})\gamma_0 \sum_{j\geq 1} \gamma^{j-1} = \mathit{\Gamma}_0^1 - \mathit{\Gamma}_0 + \gamma_0 \frac{\mathit{\Gamma}^1 - \mathit{\Gamma}}{1-\gamma}$$

$$(1-\gamma(1,y,0)) \sum_{k>j} \gamma^{k-1-j}(1,y,0) = 1$$

Bounding $L(\vartheta)$

Let $\vartheta \in \mathbb{C}$ and $H = \{\Delta \in [0,1]\} \in \mathcal{F}$, then

$$\begin{aligned}
\|L(\vartheta)\| &= \left\| \int_\Omega e^{-\vartheta \Delta_1} \, d\mathbb{P} \right\| && (1) \\
&\leq \int_\Omega \|e^{-\vartheta \Delta_1}\| \, d\mathbb{P} && (2) \\
&\leq \int_\Omega e^{-Re(\vartheta)\Delta_1} \, d\mathbb{P} && (3) \\
&\leq \int_H e^{-Re(\vartheta)\Delta_1} \, d\mathbb{P} + \int_{H^C} e^{-Re(\vartheta)\Delta_1} \, d\mathbb{P} && (4) \\
&\leq \int_H d\mathbb{P} + e^{-Re(\vartheta)} \int_{H^C} d\mathbb{P} && (5) \\
&\leq \mathbb{P}\{\Delta_1 \leq 1\} + e^{-Re(\vartheta)}(1 - \mathbb{P}\{\Delta_1 \leq 1\}) < 1 \Leftrightarrow Re(\vartheta) > 0
\end{aligned}$$

Bounding $L(\theta + \lambda - \lambda g(zl(v))$

- Assume $Re(\theta) \geq 0$, $1 \geq \|z\|$, and $Re(v) \geq 0$.
- Clearly, $Re(\theta + \lambda - \lambda g(zl(v)) > 0$ if $Re(\theta) > 0$ or $Re(g(zl(v))) < 1$.

Theorem (Schwarz Lemma)

Let $g(z)$ be an analytic function in the unit ball $B(0,1)$ with $\|g(z)\| \leq 1$ and $g(0) = 0$. Then $\|g(z)\| \leq \|z\|$ in $B(0,1)$

- $Re(g(zl(v))) \leq \|g(zl(v))\| \leq \|zl(v)\| \leq \|z\|\|l(v)\| < 1$ if

  $\|z\| < 1$ **or** $Re(v) > 0$.

Result for Φ

- Solving for $\Phi_{\mu=\nu}$ and $\Phi_{\mu>\nu}$ analogously and adding to $\Phi_{\mu<\nu}$,

$$\Phi = \mathcal{H}_{xy}^{-1}\left(\zeta_0^1 - \mathit{\Gamma}_0 + \frac{\gamma_0}{1-\gamma}\left(\zeta^1 - \mathit{\Gamma}\right)\right)(M, V)$$

where

$$\begin{aligned} \gamma &= \gamma\left(\alpha_0 \alpha x, \beta_0 + \beta + y, h_0 + h\right) \\ \mathit{\Gamma} &= \gamma\left(\alpha x, \beta + y, h\right) \\ \zeta^1 &= \gamma(\alpha x, \beta, h) \end{aligned}$$

Results for a Special Case

To demonstrate that tractable results can be derived from this, we will consider a special case where

- $\Delta_k \in [\text{Exponential}(\mu)] \implies L(\theta) = \frac{\mu}{\mu+\theta}$
- $n_k \in [\text{Geometric}(a)] \implies g(z) = \frac{az}{1-bz}, (b = 1 - a)$
- $w_{jk} \in [\text{Exponential}(\xi)] \implies l(u) = \frac{\xi}{\xi+u}$
- $(N_0, W_0) = (0, 0)$

Recall $\gamma\left(z, v, \theta\right) = L\left[\theta + \lambda - \lambda g\left(zl\left(v\right)\right)\right]$

Theorem 1

$$
\begin{aligned}
\Phi &= \Phi\left(1, z, 0, v, 0, \theta\right) = \mathbb{E}\left[z^{N_\rho} e^{-vW_\rho} e^{-\theta\tau_\rho}\right] \\
&= 1 - \left(1 - \frac{\mu}{\mu+\theta+\lambda}\frac{v+\xi(1-bz)}{v+\xi(1-c_2 z)}\right) \\
&\quad \times \left(1 + \frac{b\mu}{\lambda+b\theta} + \frac{a\lambda\mu}{\left(\lambda+b\theta\right)\left(\lambda+\theta\right)}\phi\left(z, v, \theta\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\phi(z, v, \theta) &= \frac{v+\xi}{v+\xi(1-c_1 z)} - \frac{c_1 z\xi \boxed{Q(M-1, c_1 z\xi V)} e^{-(v+\xi(1-c_1 z))V}}{v+\xi(1-c_1 z)} \\
&\quad - \frac{(c_1 z\xi)^M \boxed{P(M-1, (\xi+v)V)}}{(v+\xi(1-c_1 z))(\xi+v)^{M-1}},
\end{aligned}
$$

$$
c_1 = \frac{\lambda+b\theta}{\lambda+\theta}, \qquad c_2 = \frac{\lambda+b(\mu+\theta)}{\lambda+\mu+\theta},
$$

and $Q(x,y) = \frac{\Gamma(x,y)}{\Gamma(x)}$ is the lower regularized gamma function.

Corollaries: Marginal Transforms

$$\Phi(1,z,0,0,0,0) = \mathbb{E}\left[z^{N_\rho}\right]$$
$$= \frac{zQ(M-1,z\xi V)e^{-\xi(1-z)V} + z^M P(M-1,\xi V)}{\mu+\lambda-(\lambda+b\mu)z}$$
$$\Phi(1,1,0,v,0,0) = \mathbb{E}\left[e^{-vW_\rho}\right]$$
$$= \frac{\lambda v + b\mu v + a\xi\mu\phi(1,v,0)}{a\xi\mu + (\lambda+\mu)v}$$
$$\Phi(1,1,0,0,0,\theta) = \mathbb{E}\left[e^{-\theta\tau_\rho}\right]$$
$$= 1 - \frac{\theta}{\mu+\theta}\left[1 + \frac{b\mu}{\lambda+b\theta} + \frac{a\lambda\mu\phi(1,0,\theta)}{(\lambda+b\theta)(\lambda+\theta)}\right]$$

Useful Results: CDF of Observed Passage Time, τ_ρ

$$F_{\tau_\rho}(\vartheta) = \mathbb{P}\{\tau_\rho < \vartheta\}$$
$$= \lambda P(M-1, \xi V) \sum_{j=0}^{M-1} c_j \phi_j(\vartheta) + \lambda e^{-\xi\lambda} \sum_{k=0}^{M-2} \frac{(\xi V)^k}{k!} \sum_{j=0}^{k} d_j \phi_j(\vartheta)$$

where

$$c_j = \binom{M-1}{j} (a\lambda)^j b^{M-1-j}$$
$$d_j = \binom{k}{j} (a\lambda)^j b^{k-j}$$
$$\phi_j(\vartheta) = \frac{1}{\lambda^{j+1}} P(j+1, \lambda\vartheta) - \frac{e^{-\mu\vartheta}}{(\lambda-\mu)^{j+1}} P(j+1, (\lambda-\mu)\vartheta)$$

Useful Results: Means at τ_ρ

- $\mathbb{E}\left[N_\rho\right] = \frac{\lambda+b\mu}{a\mu} + M - (M-1)Q(M-1, \xi V) + \xi V Q(M-2, \xi V)$
- $\mathbb{E}\left[W_\rho\right] = \frac{\mathbb{E}[N_\rho]}{\xi} = \mathbb{E}\left[N_\rho\right]\mathbb{E}[w_{11}]$

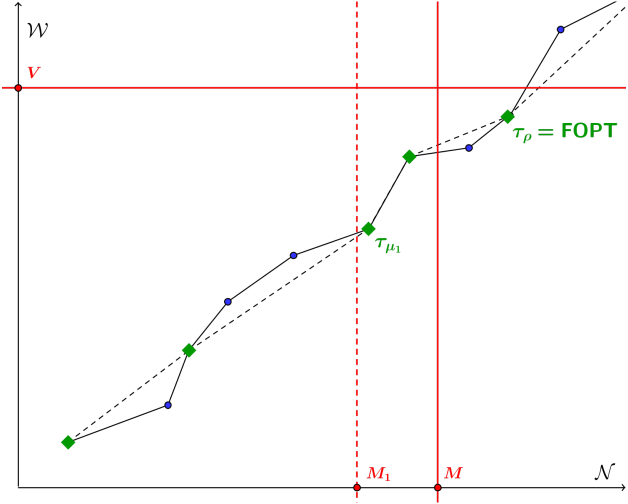
Simulation

We simulated 1,000 realizations of the process under each of the some sets of parameters $(\lambda, \mu, a, \xi, M, V)$ and recorded the sample means:

Parameters	$\mathbb{E}[N_\rho]$	S. Mean	Error	$\mathbb{E}[W_\rho]$	S. Mean	Error
(1)	989.08	988.82	0.26	989.08	990.06	0.98
(2)	990.63	990.39	0.24	990.63	990.27	0.36
(3)	989.28	989.92	0.64	989.28	988.97	0.31
(4)	989.08	989.08	0.00	989.08	989.68	0.31
(5)	503.00	502.73	0.27	1006.00	1005.04	0.96
(6)	1002.00	1001.57	0.43	501.00	500.91	0.09
(7)	803.00	802.68	0.32	803.00	802.67	0.33
(8)	752.00	752.10	0.10	752.00	751.68	0.32
(9)	493.57	493.46	0.11	987.14	986.59	0.55

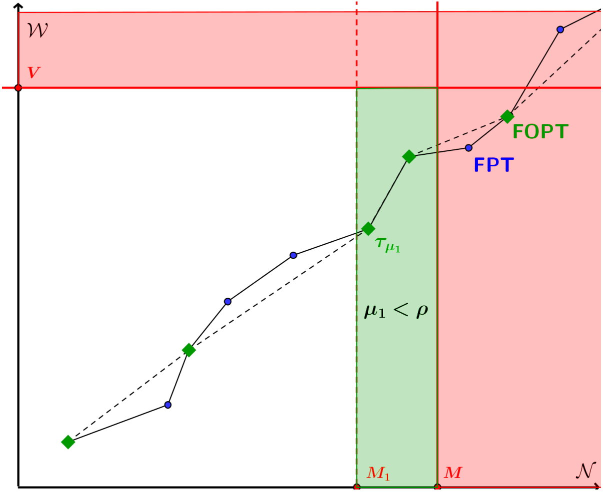
Auxiliary Threshold Model

We introduce another threshold $M_1 < M$ with $\mu_1 = \min\{n : N_n > M_1\}$



Auxiliary Threshold Model

We will be concerned with the confined process where $\mu_1 < \rho$,



Functional of Interest

Our functional of interest will be similar to before, but containing terms associated with the auxiliary crossing at μ_1

$$\Phi_{\mu_1<\rho} = \Phi_{\mu_1<\rho}\left(z_0, z, \alpha_0, \alpha, v_0, v, \beta_0, \beta, \theta_0, \theta, h_0, h\right)$$
$$= \mathbb{E}\Bigg[\boxed{z_0^{N_{\mu_1-1}} z^{N_{\mu_1}}}\alpha_0^{N_{\rho-1}}\alpha^{N_\rho}\boxed{e^{-v_0 W_{\mu_1-1}-vW_{\mu_1}}}e^{-\beta_0 W_{\rho-1}-\beta W_\rho}$$
$$\times \boxed{e^{-\theta_0\tau_{\mu_1-1}-\theta\tau_{\mu_1}}}e^{-h_0\tau_{\rho-1}-h\tau_\rho}\boxed{\mathbf{1}_{\{\mu_1<\rho\}}}\Bigg]$$

A new capability is to find

$$\Phi_{\mu_1>\rho}\left(1,1,1,1,0,0,0,0,0,-h,0,h\right) = \mathbb{E}\left[e^{-h\left(\tau_\rho-\tau_{\mu_1}\right)}\right]$$

Strategy and Model 1: Constant Observation

We use an operator $\mathcal{H}_{\mu_1} = \mathcal{LC}_s \circ D_q \circ D_p$ adaped to work with the additional discrete threshold, but the path to results remains the same

$$\Phi_{\mu_1<\rho} \xrightarrow{\mathcal{H}_{\mu_1}} \Psi_{\mu_1<\rho} \xrightarrow{\text{Assumptions}} \Psi_{\mu_1<\rho} \text{ (convenient) } \xrightarrow{\mathcal{H}_{\mu_1}^{-1}} \Phi_{\mu_1<\rho} \text{ (tractable)}$$

In this model, we have

- $\Delta_k = c \; a.s.$
- n_k with **arbitrary** finite distribution $(p_1, p_2, ..., p_m)$
- $w_{jk} \in [\text{Gamma}(\alpha, \xi)]$

Theorem

$$\Phi_{\mu_1<\rho}(1,z,1,1,0,v,0,0,0,\theta,0,0) = \mathbb{E}\left[z^{N_{\mu_1}} e^{-vW_{\mu_1}} e^{-\theta\tau_{\mu_1}} \mathbf{1}_{\{\mu_1<\rho\}}\right]$$

$$= \left\{ \sum_{k=0}^{M_1-1} z^k F_k \sum_{m=0}^{M-1-k} z^m E_m \left(\frac{\xi}{v+\xi}\right)^{\alpha(k+m)} P(\alpha(k+m),(v+\xi)V) \right.$$

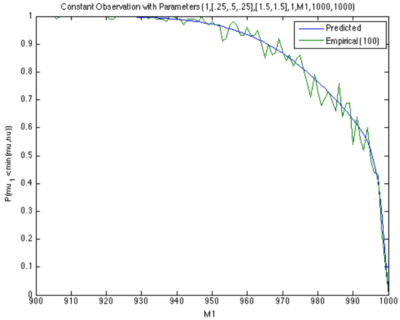
$$\left. - \sum_{k=0}^{M_1-1} z^k \left(\frac{\xi}{v+\xi}\right)^{\alpha k} P(\alpha k,(v+\xi)V) \sum_{n=0}^{k} E_n F_{k-n} \right\} e^{-c(\theta+\lambda)}$$

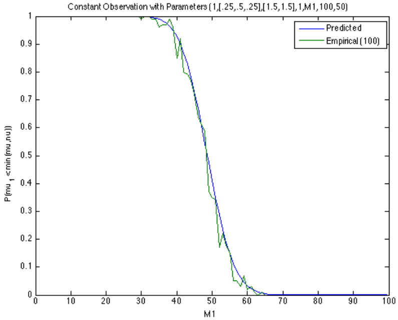
$$F_j = \sum_{r=0}^{\left\lfloor \frac{R-1}{R} j \right\rfloor} (c\lambda)^{j-r} \boxed{Li_{-(j-r)}\left(e^{-c(\theta+\lambda)}\right)} \sum_{\substack{\|\boldsymbol{\beta}\|_1=j \\ [R]\cdot\boldsymbol{\beta}=r+j}} \frac{p_1^{\beta_1}\cdots p_R^{\beta_R}}{\beta_1!\ \cdots\ \beta_R!},$$

$$E_j = \sum_{r=0}^{\left\lfloor \frac{R-1}{R} j \right\rfloor} (c\lambda)^{j-r} \sum_{\substack{\|\boldsymbol{\beta}\|_1=j \\ [R]\cdot\boldsymbol{\beta}=r+j}} \frac{p_1^{\beta_1}\cdots p_R^{\beta_R}}{\beta_1!\ \cdots\ \beta_R!}$$

Results and Simulation

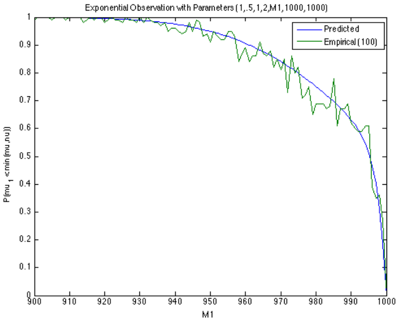
We find $\Phi_{\mu_1<\rho}(1,1,1,1,0,0,0,0,0,0,0,0) = \mathbb{E}\left[\mathbf{1}_{\{\mu_1<\rho\}}\right] = \mathbb{P}\{\mu_1 < \rho\}$ and compare to simulated results for a range of M_1 values

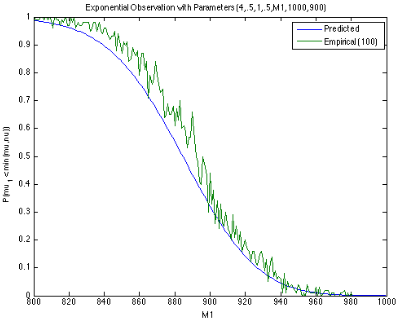




Simulation for an Alternate Model

We did the same under the assumptions $\Delta_1 \in$ [Exponential(μ)], $n_1 \in$ [Geometric(a)], $w_{11} \in$ [Exponential(ξ)].





Extensions and Future Work

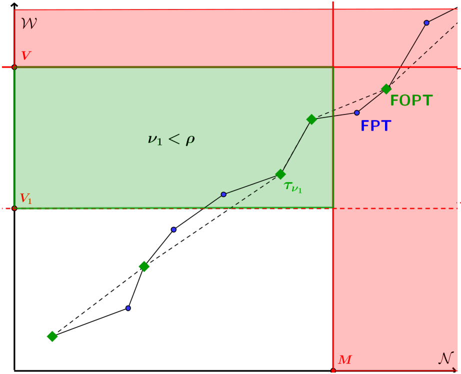


Figure: Continuous Auxiliary Model, $\nu_1 = \min\{n : W_n > V_1\}$

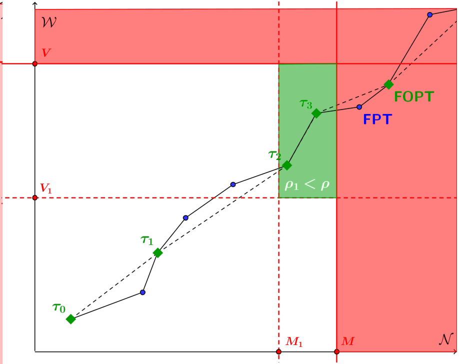
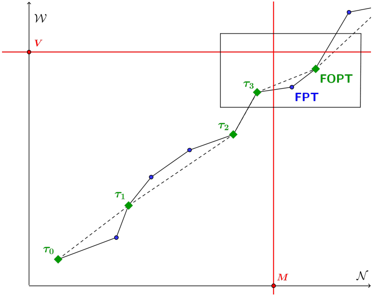


Figure: Dual Auxiliary Model, $\rho_1 = \max\{\mu_1, \nu_1\}$

Extensions and Future Work, Time-Sensitive Analysis

- Time-sensitive models
  - We try to "interpolate" the process in random vicinities of the FOPT.



  - New strategies: generalization to ISI processes viewed upon stopping times, additional operators, convolutions
  - New types of results: $\mathbb{P}\{N_\rho = k, \tau_\rho < t\}$, $\mathbb{P}\{W_\rho < \vartheta, \tau_{\rho-1} < t < \tau_\rho\}$

Extensions and Future Work - n-Dimensional Model

- n-component model
  - n-dimensional process
  - Threshold(s) on each of n components (d stopping times for observed crossings)
  - New strategies: Additional operators, generalization of the confined process strategy, new computational techniques
  - New types of results: $\mathbb{P}\{\mu_3 < \mu_5 < \mu_2\}$, $\mathbb{P}\{\mu_1 = \mu_3, \tau_{\min\{\mu_1,\ldots,\mu_n\}} < t\}$