Calculus 1 - Oct 11 Lecture

4.1 Maximum and Minimum Values

Upcoming Dates

Oct 16 - 4.4 homework due

Oct 17 - Test 2 (does **not** include 4.1)

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Office hours: 11-12 M, 10-12 WF (Crawford 111)

Feel free to e-mail/stop by for help this week!

Extreme Values of a Function

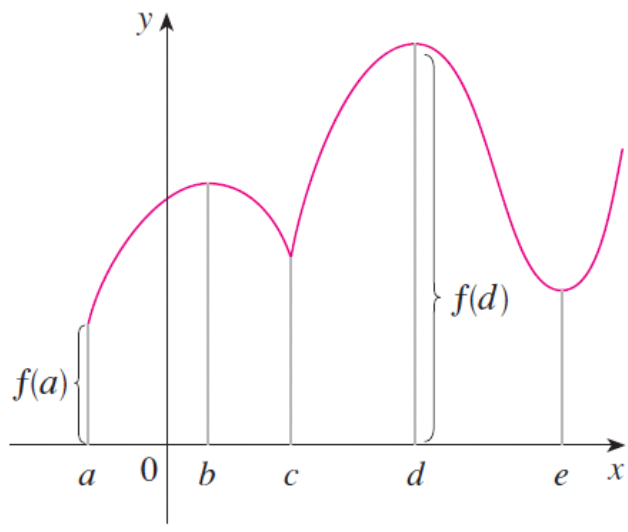
Some of the most important applications of differential calculus are *optimization problems*, in which we are required to find the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter:

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration.)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?
- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to finding the maximum or minimum values of a function. Let's first explain exactly what we mean by maximum and minimum values.

$f: D \to \mathbb{R}$

1 DEFINITION A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geqslant f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the **maximum value** of f on D. Similarly, f has an **absolute minimum** at c if $f(c) \leqslant f(x)$ for all x in D and the number $f(c)$ is called the **minimum value** of f on D. The maximum and minimum values of f are called the **extreme values** of f.



2 DEFINITION A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geqslant f(x)$ when x is near c. [This means that $f(c) \geqslant f(x)$ for all x in some open interval containing c.] Similarly, f has a **local minimum** at c if $f(c) \leqslant f(x)$ when x is near c.

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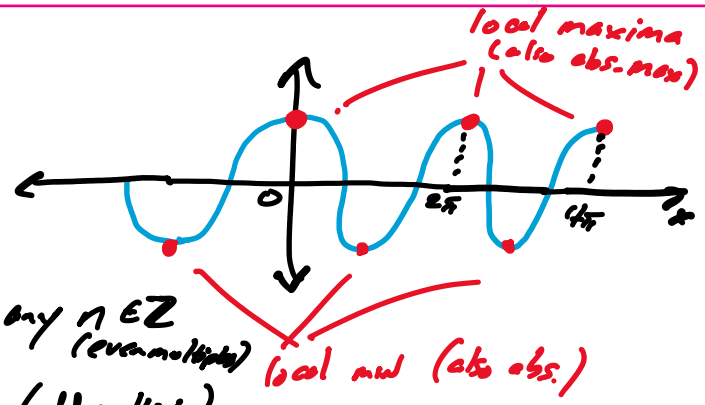
<u>Example</u> $f(x) = \cos x$

$f: \mathbb{R} \to [-1, 1]$



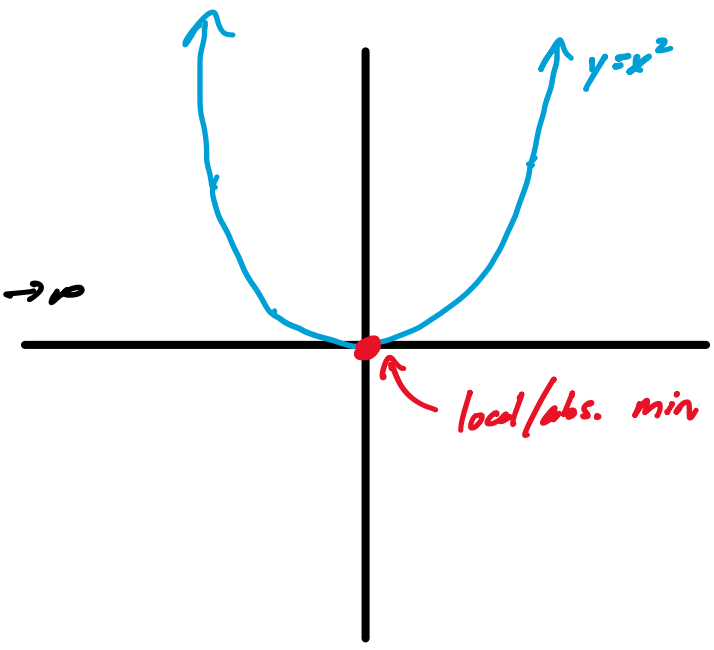
$\cos x = 1$ if $x = 2\pi n$ for any $n \in \mathbb{Z}$ (even multiples)

$= -1$ if $x = (2n+1)\pi$ (odd multiples)

<u>Example</u> $f(x) = x^2$

min at $x=0$, $f(0)=0$

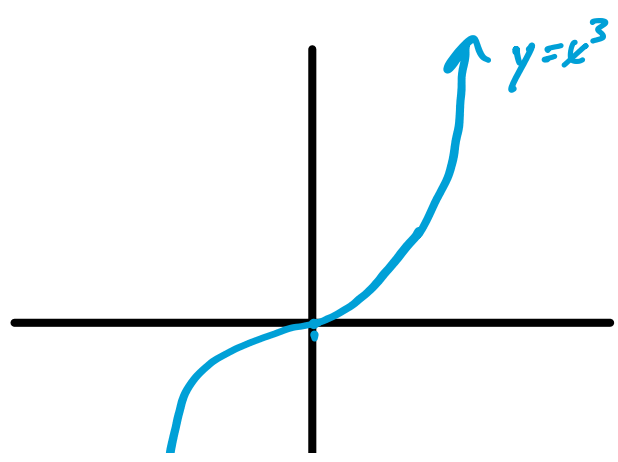
no max exists since $f(x) \to \infty$ as $x \to \infty$.



<u>Example</u> $f(x) = x^3$

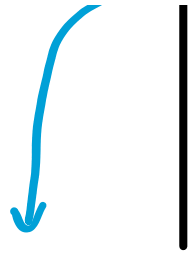
No max, no min since $\lim_{x\to\infty} f(x) = \infty$

and $\lim f(x) = -\infty$



$x \to \infty$

and $\lim_{x \to -\infty} f(x) = -\infty$



Example

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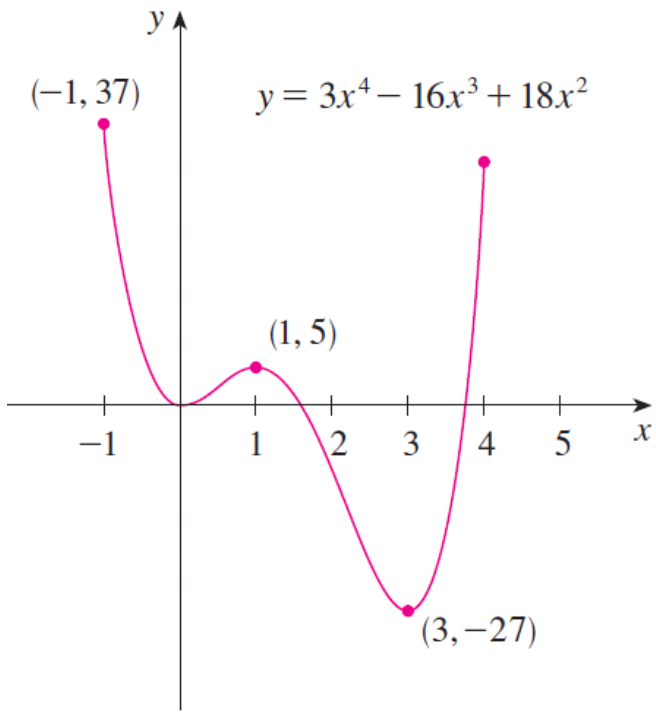
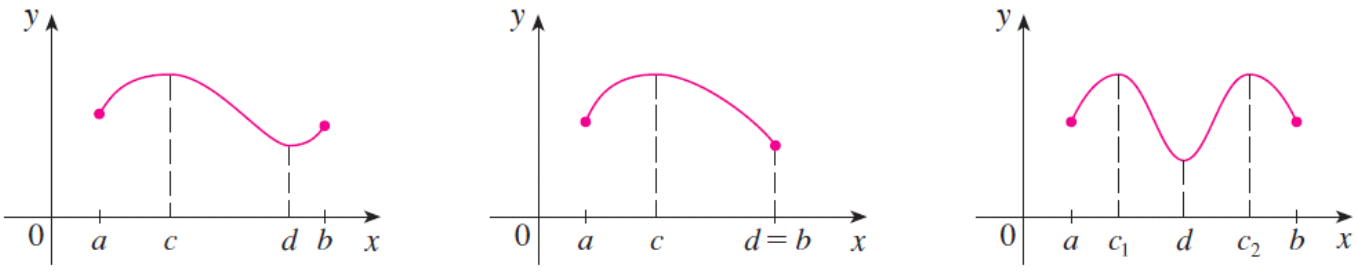
$f(x) = 3x^4 - 16x^3 + 18x^2, \quad -1 \le x \le 4$



Existence of Extreme Values

3 THE EXTREME VALUE THEOREM If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

The Extreme Value Theorem is illustrated in Figure 5. Note that an extreme value can be taken on more than once. Although the Extreme Value Theorem is intuitively very plausible, it is difficult to prove and so we omit the proof.



Figures 6 and 7 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

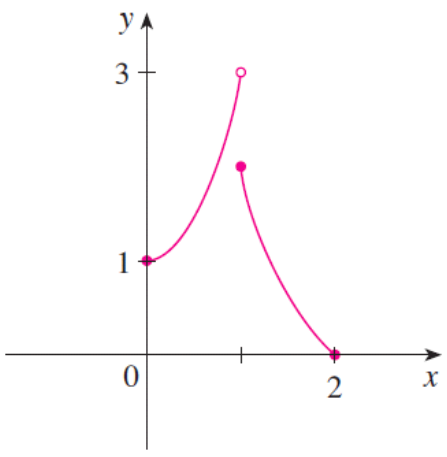


FIGURE 6
This function has minimum value $f(2) = 0$, but no maximum value.

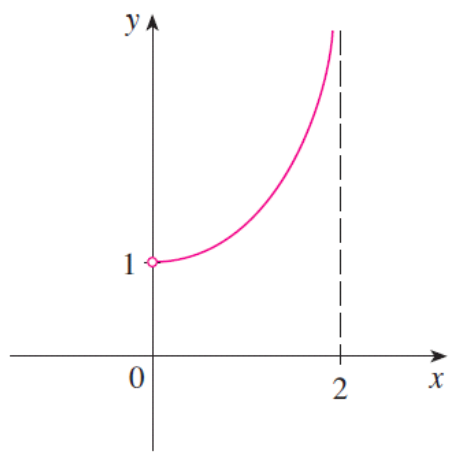
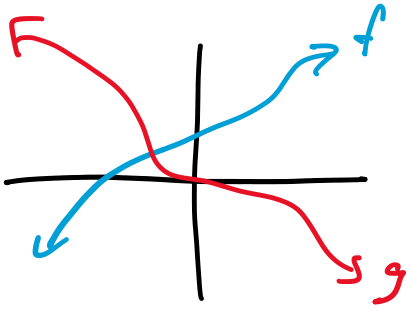


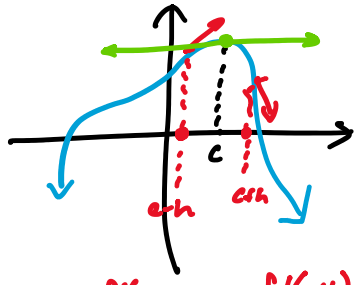
FIGURE 7
This continuous function g has no maximum or minimum.

Finding Extreme Values



$f'(x) > 0$

$g'(x) < 0$



$f'(c+h) > 0 \quad f'(c+h) < 0$

$f'(c)$ appears to be 0 at the max.

Is this always true?

Recall: $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ if it exists

Suppose f has a local max. at $x = c$. Then $f(c) \geq f(x)$ for all x within some interval containing c. For some $h > 0$,

$$f(c) \geq f(c+h)$$

$$0 \geq f(c+h) - f(c)$$

If $h < 0$...

$$0 \geq \frac{f(c+h) - f(c)}{h}$$

$$0 \leq \frac{f(c+h) - f(c)}{h}$$

$$\lim_{h \to 0^+} 0 \geq \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

if f is continuous

left-sided limit implies

$$0 \geq f'(c)$$

if $f'(c)$ exists

$$0 \leq f'(c)$$

Thus, $f'(c) = 0$.

Fermat's Theorem If f has a local extreme value at c,

Fermat's Theorem If f has a local extreme value at c, then ~~$f'(c)=0$ if it exists~~ c is a critical number of f.

Def: A <u>critical number</u> of f is a number c in its domain such that $f'(c)=0$ or $f'(c)$ does not exist

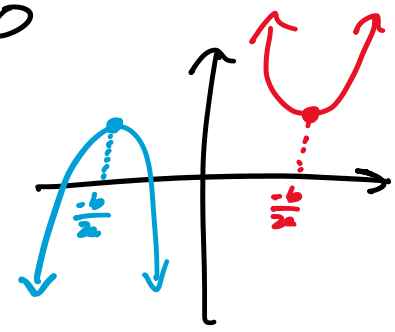
Ex $f(x) = ax^2 + bx + c, \quad a \neq 0$

$$f'(x) = 2ax + b = 0$$

$$2ax = -b$$

$$x = -\frac{b}{2a}$$

one crit. value at the vertex. it is a max or min.

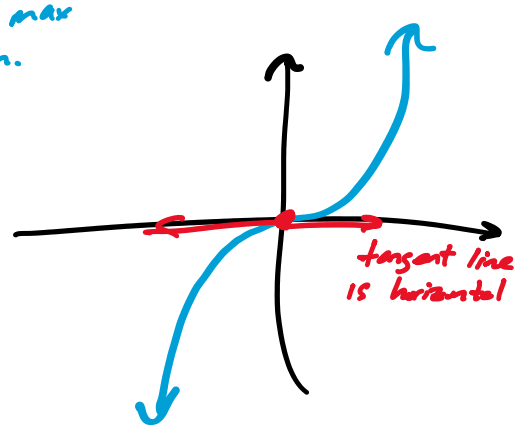


Ex $f(x) = x^3$

$$f'(x) = 3x^2 = 0$$

$$x = 0$$

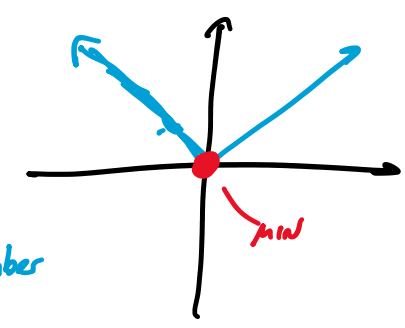
One crit. number, but it is not an extreme value



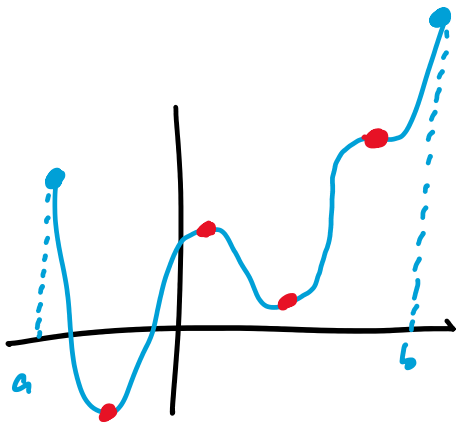
Ex $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$$f'(x) = \begin{cases} 1, & \text{if } x > 0 \\ \text{DNE}, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

critical number (minimum)



Problem: Find the absolute (global) extrema of a continuous function f on a closed interval $[a,b]$.



• critical numbers

Closed Interval Method

1. Find $f(c)$ for critical numbers s.t. $a<c<b$.
2. Find $f(a)$ and $f(b)$.
3. The largest output from 1 & 2 is the absolute max.

Example

$f(x) = x^3 - 3x^2 + 1$ $\qquad$ $\frac{-1}{2} \le x \le 4$

Find the abs. max + min on this interval.

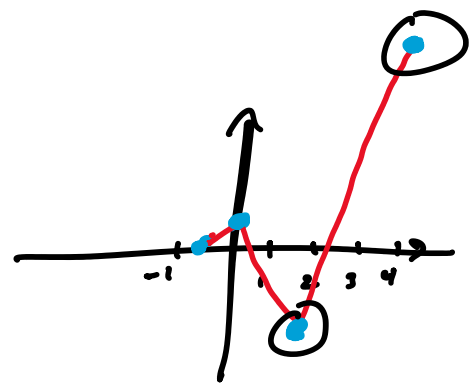
① $f'(x) = 3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0, 2$

$f(0) = 0^3 - 3(0)^2 + 1 = 1$

$f(2) = 2^3 - 3(2)^2 + 1 = 8 - 12 + 1 = -3$



② $f(\frac{-1}{2}) = (\frac{-1}{2})^3 - 3(\frac{-1}{2})^2 + 1 = \frac{1}{8} - \frac{3}{4} + 1 = \frac{3}{8}$

$f(4) = 4^3 - 3(4)^2 + 1 = 64 - 36 + 1 = 29$

③ Abs max: $(4, 29)$ $\qquad$ Abs min: $(2, -3)$

Example

EXAMPLE 10 The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ s, is given by

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

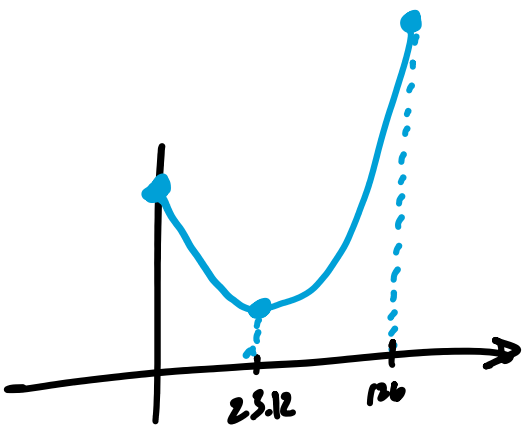
(in feet per second). Using this model, estimate the absolute maximum and minimum values of the *acceleration* of the shuttle between liftoff and the jettisoning of the boosters.

$$a(t) = 0.003906t^2 - 0.18058t + 23.61 \quad = v'(t)$$

$$a'(t) = 0.007812t - 0.18058 = 0$$

$$0.007812t = 0.18058$$

$$t = 23.12 \text{ s}$$



$a(0) = 23.61$ ft/s²

$a(23.116) = 21.52$ ft/s² ← min.

$a(126) = 62.87$ ft/s² ← max