Time Sensitive Analysis of d-dim ISI Processes

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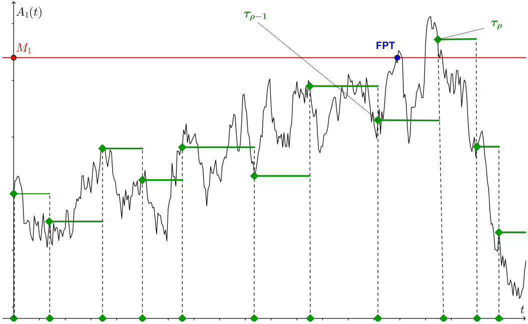
ISI Processes and a Standard Problem

- $A(t) = (A_1(t), \cdots A_d(t))$ is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$
- $A(t)$ is a **independent and stationary increment** (ISI) jump process
- We consider the **exit** of $A(t)$ from $\mathcal{R} = (-\infty, M_1) \times \cdots \times (-\infty, M_d)$
  - i.e. when any $A_j(t)$ crosses threshold M_j
- The **first passage time** (FPT) is $T_\alpha = \inf\{t : A(t) \notin \mathcal{R}\}$
- We seek probabilistic data about the FPT T_α, position of the process at exit $A(T_\alpha)$, marginal values $A_j(T_\alpha)$, the relationships between exits of each component, etc.

Delayed Observation and a Complication

- $A(t)$ is inaccessible to the observer
- Information is only available upon an independent renewal process $\{\tau_n\}$
  - We can only analyze the pw-constant $A_n = A(\tau_n)$
  - The FPT is not accessible
- We have the **virtual passage time** τ_ρ for $\rho = \inf\{n : A(\tau_n) \notin \mathcal{R}\}$

Figure: A 1D process under delayed observation



Previous Relevant Results

- "Time insensitive" joint transforms[4,5]

$$\mathbb{E}\left[e^{-iv_1\cdot A_{\rho-1}-iv_2\cdot A_\rho-h_0\tau_{\rho-1}-h\tau_\rho}\right]$$

  - Marginal transforms: $\mathbb{E}\left[e^{-iv_{11}(A_1)_{\rho-1}}\right]$, $\mathbb{E}\left[e^{-iv_{13}(A_3)_\rho}\right]$, $\mathbb{E}\left[e^{-ih\tau_\rho}\right]$
  - Moments and distributions: $\mathbb{E}[\tau_\rho]$, $P\{\tau_\rho < t\}$, $Var((A_1)_{\rho-1})$
  - Applications to stochastic networks[4,5], stochastic games[3], queueing

- "Time sensitive" transforms in 1D[6]

$$\mathbb{E}\left[e^{-iv_1 A_{\rho-1}-iv_2 A_\rho-iwA(t)-ix_1\tau_{\rho-1}-ix_2\tau_\rho}\mathbf{1}_{\{\tau_{\rho-1}\le t<\tau_\rho\}}\right]$$

  - Marginal transforms, including $\mathbb{E}\left[e^{-iwA(t)}\mathbf{1}_{\{\tau_{\rho-1}\le t<\tau_\rho\}}\right]$
  - Joint distributions: $\mathbb{P}\{A_\rho = k, \tau_\rho < t\}$, $\mathbb{P}\{A_\rho < s, \tau_\rho < t\}$
  - Applications to finance[3], queueing[1], stochastic networks[6]

Approach

1. Introduce a new point process, $\mathcal{T} = \{T_0, T_1, \dots\}$, independent of the filtration $(\mathcal{F}_t)$, where each $\Delta_n = T_n - T_{n-1} > 0$ is independent

2. Derive a general result for a joint characteristic function of the ISI processes

$$F_n(t, v_0, \dots, v_m, w, x) = \mathbb{E}\left[e^{-i\sum_{j=0}^{m} v_j \cdot A(T_j) - iw \cdot A(t) - ix \cdot \Delta} \mathbf{1}_{\{T_{n-1} \le t < T_n\}}\right]$$

3. Apply the result to $A(t)$ as a compound Poisson process

4. Embed the $\{T_n\}$ into $\{\tau_n\}$ and find joint functionals restricted to random intervals $I_0 = [0, \tau_{\rho-1})$ and $I_1 = [\tau_{\rho-1}, \tau_\rho)$

Result for General ISI Processes

Theorem 1.

For the d-dimensional ISI process $A(t)$ on the trace σ-algebra $\mathcal{F} \cap \{T_{n-1} \le t < T_n\}$ where T_{n-1} and Δ_n are independent of $\mathcal{F}_t$ satisfies

$$F_n^* (\theta, v_0, ..., v_m, w, x)$$

$$= \prod_{j=0}^{n-1} \gamma_j (b_j + w, x_j + \theta) E \left[e^{-ix_n \Delta_n} \psi (b_n, b_n + w, \Delta_n, \theta) \right] \prod_{j=n+1}^{m} \gamma_j (b_j, x_j)$$

under the notation

$$b_j = \sum_{i=j}^{m} v_i, \qquad \varphi (b, s) = E \left[e^{-ib \cdot A(s)} \right],$$

$$\psi (b, x, r, \theta) = \left(e^{-i\theta(\cdot)} \varphi (b, \cdot) \right) * \varphi (x, \cdot) (r)$$

$$\gamma_j (a, \vartheta) = E \left[e^{-i\vartheta \Delta_j} e^{-ia \cdot \left[A\left(T_j\right) - A\left(T_{j-1}\right) \right]} \right] = E \left[e^{-ia \cdot A\left(\Delta_j\right) - i\vartheta \Delta_j} \right]$$

where $v_j, w \in \mathbb{C}_-^d$ and $x \in \mathbb{C}_-^{m+1}$

Outline of Theorem 1 Proof (1)

- We seek $F_n^*\left(\theta, v_0, ..., v_m, w, x\right) = \int\limits_{-\infty}^{\infty} e^{-i\theta t} F_n\left(t, v_0, ..., v_m, w, x\right)\, dt$
- Let $r = (r_0, ..., r_m)$ and $s_n = \sum_{j=0}^{n} r_j$.
- Writing F_n as an integral w.r.t. joint distribution of $(\Delta_0, ..., \Delta_m)$ and rewriting $A(T_k)$ and $A(t)$,

$$
\begin{aligned}
&F_n^*\left(\theta, v_0, ..., v_m, w, x\right) \\
&= \int_{t\in\mathbb{R}} e^{-i\theta t} \int_{r\in\mathbb{R}^{m+1}} e^{-ix\cdot r} E\Bigg[e^{-i\sum_{j=0}^{n-1}\left(b_j+w\right)\cdot\left[A\left(s_j\right)-A\left(s_{j-1}\right)\right]-i\left(b_n+w\right)\cdot\left[A(t)-A\left(s_{n-1}\right)\right]} \\
&\qquad \times e^{-ib_n\cdot\left[A(s_n)-A(t)\right]-i\sum_{j=n+1}^{m} b_j\cdot\left[A\left(s_j\right)-A\left(s_{j-1}\right)\right]} \\
&\qquad \times \mathbf{1}_{\left\{s_{n-1}\le t< s_{n-1}+r_n\right\}}\Bigg] dP_{\bigotimes_{j=0}^{m}\Delta_j}\left(r_0, ..., r_m\right)\, dt
\end{aligned}
$$

Outline of Theorem 1 Proof (2)

- By the ISI properties,

$$
\begin{aligned}
&F_n^*(\theta, v_0, \dots, v_m, w, x) \\
&= \int_{t\in\mathbb{R}} e^{-i\theta t} \int_{\mathbf{r}\in\mathbb{R}^{m+1}} e^{-ix\cdot\mathbf{r}} \prod_{j=0}^{n-1} \varphi(b_j + w, r_j)\, \varphi(b_n + w, t - s_{n-1}) \\
&\times \varphi(b_n, s_{n-1} + r_n - t) \prod_{j=n+1}^{m} \varphi(b_j, r_j)\, \mathbf{1}_{\{s_{n-1}\le t< s_{n-1}+r_n\}}\, dP_{\bigotimes_{j=0}^{m}\Delta_j}(r_0, \dots, r_m)\, dt
\end{aligned}
$$

- By Fubini's Theorem and the independence of $\Delta_0, \dots, \Delta_m$,

$$
\begin{aligned}
&F_n^*(\theta, v_0, \dots, v_m, w, x) \\
&= \prod_{j=0}^{n-1} \int_{r_j\in\mathbb{R}} e^{-i(x_j+\theta)r_j} \varphi(b_j + w, r_j)\, dP_{\Delta_j}(r_j) \prod_{j=n+1}^{m} \int_{r_j\in\mathbb{R}} e^{-ix_j r_j} \varphi(b_j, r_j)\, dP_{\Delta_j}(r_j) \\
&\quad \times \int_{r_n\in\mathbb{R}} e^{-ix_n r_n} \int_{t=s_{n-1}}^{s_{n-1}+r_n} e^{-i\theta(t-s_{n-1})} \varphi(b_n + w, t - s_{n-1}) \\
&\qquad\qquad\qquad \times \varphi(b_n, s_{n-1} + r_n - t)\, dt\, dP_{\Delta_n}(r_n)
\end{aligned}
$$

Outline of Theorem 1 Proof (3)

- For the t-independent integrals, return to expectation form.
- By the translation invariance property of the Lebesgue measure and taking $u = t - s_{n-1}$, we can adjust the t-integral to an associated u-integral,

$$
\begin{aligned}
&F_n^* \left(\theta, v_0, ..., v_m, w, x\right) \\
&= \prod_{j=0}^{n-1} \gamma_j \left(b_j + w, x_j + \theta\right) \prod_{j=n+1}^{m} \gamma_j (b_j, x_j) \\
&\quad \times \int_{r_n \in \mathbb{R}} e^{-i x_n r_n} \int_{u=0}^{r_n} e^{-i\theta u} \varphi\left(b_n + w, u\right) \varphi\left(b_n, r_n - u\right) du \, dP_{\Delta_n}\left(r_n\right) \\
&= \prod_{j=0}^{n-1} \gamma_j \left(b_j + w, x_j + \theta\right) E\left[e^{-i x_n \Delta_n} \psi\left(b_n + w, b_n, \Delta_n, \theta\right)\right] \prod_{j=n+1}^{m} \gamma_j \left(b_j, x_j\right)
\end{aligned}
$$

$\square$

Application of Theorem 1 to a Compound Poisson Process

- Let $A(t) = \sum_{k \geq 0} a_k \varepsilon_{t_k}([0,t])$ be a compound Poisson process in $\mathbb{R}^d$
  - Characteristic function of marks - $g(v) = \mathbb{E}\left[e^{-iv \cdot a_1}\right]$

Theorem 2.

Let $a(t)$ be a d-dimensional compound Poisson process of rate λ and assume $\mathcal{T}$ is independent of $\mathcal{F}_t$, then

$$
\begin{aligned}
&F_n^*\left(\theta, v_0, ..., v_m, w, x\right) \\
&= \prod_{j=0}^{n-1} L_j(\theta + x_j - i\lambda(1 - g(b_j + w))) \prod_{j=n+1}^{m} L_j(x_j - i\lambda(1 - g(b_j))) \\
&\quad \times \frac{L_n(x_n - i\lambda(1 - g(b_n))) - L_n(\theta + x_n - i\lambda(1 - g(b_n + w)))}{i\theta + \lambda(g(b_n + w) - g(b_n))}
\end{aligned}
$$

where $L_j(z) = E\left[e^{-iz\Delta_j}\right]$ denotes the characteristic function of Δ_j.

Proof of Theorem 2

- Since A is a compound Poisson process,

$$\varphi(a,s) = E\left[e^{-ia\cdot A(s)}\right] = e^{-\lambda s(1-g(a))},$$
$$\gamma_j\left(a,\vartheta\right) = E\left[e^{-ia\cdot A\left(\Delta_j\right)-i\vartheta\Delta_j}\right] = L_j\left(\vartheta - i\lambda(1-g(a)\right),$$

- In addition,

$$\psi\left(b,a,\Delta_n,\theta\right) = \int_0^{\Delta_n} e^{-i\theta t}\varphi\left(b,t\right)\varphi\left(a,\Delta_n - t\right)\,dt$$
$$= \frac{e^{-\lambda(1-g(a))\Delta_n} - e^{-(i\theta+\lambda(1-g(b)))\Delta_n}}{i\theta + \lambda\left(g\left(a\right) - g\left(b\right)\right)}$$

Random Vicinities of T_α

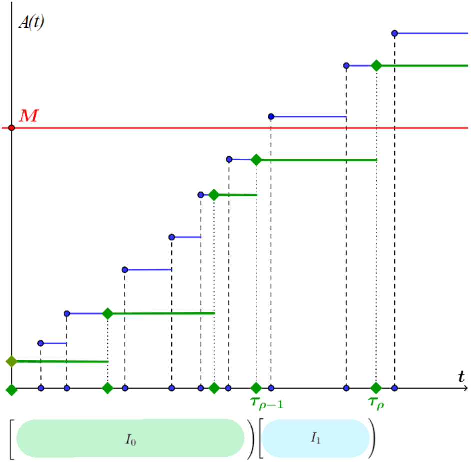


Figure: 1D compound Poisson process with random intervals in the vicinity of T_α

- For discussion, let $d = 2$.
- Theorems 1-2 required the independence of $\{T_n\}$ from the filtration
- We cannot replace T_n's with $\tau_{\rho-1}$ and τ_ρ as they fail this independence
- We can **embed** $\{T_n\}$ into $\{\tau_n\}$:

$$T_0 = 0, \quad T_1 = \tau_{j-1}, \quad T_2 = \tau_j, \quad T_3 = \tau_{k-1}, \quad T_4 = \tau_k,$$

- Then, Theorem 2 determines

$$F^*_{j=\nu_1<\nu_2=k}(\theta, v_1, v_2, w, h_0, h)$$
$$= \int_{t\geq 0} e^{-i\theta t} E\left[e^{-iv_1\cdot A_{j-1}-iv_2\cdot A_j-iw\cdot A(t)-ih_0\tau_{j-1}-ih\delta_j}\mathbf{1}_{\{\tau_{j-1}\leq t<\tau_j\}}\boxed{\mathbf{1}_{\{\nu_1=j,\nu_2=k\}}}\right] d$$

where $\nu_j = \inf\{n : A_j(\tau_n) > M_j\}$, and $\delta_j = \tau_j - \tau_{j-1}$.

Partitioning and Geometric Series

- Splitting the goal functional into three parts,

$$\begin{aligned}
&\Phi^{(2)}(\theta, v_1, v_2, w, h_0, h) \\
&= \int_{t\geq 0} e^{-i\theta t} E\left[e^{-iv_1\cdot A_{\rho-1} - iv_2\cdot A_\rho - iw\cdot A(t) - ih_0\tau_{\rho-1} - ih\delta_\rho} \mathbf{1}_{\{\tau_{\rho-1}\leq t<\tau_\rho\}}\right] dt \\
&= F^*_{\nu_1<\nu_2} + F^*_{\nu_1=\nu_2} + F^*_{\nu_1>\nu_2}
\end{aligned}$$

- Each further reduce to series of results that can be derived via Theorem 2:

$$\begin{aligned}
F^*_{\nu_1<\nu_2} &= \sum_{j\geq 0}\sum_{k>j} F^*_{j=\nu_1<\nu_2=k} \\
F^*_{\nu_1=\nu_2} &= \sum_{j\geq 0} F^*_{j=\nu_1=\nu_2=k} \\
F^*_{\nu_1>\nu_2} &= \sum_{k\geq 0}\sum_{j>k} F^*_{j=\nu_1>\nu_2=k}
\end{aligned}$$

- These sum as geometric series (using independent increments, iid inter-observation times δ_j, and boundedness in the unit ball via Schwarz Lemma)

- The result is established via the series convergence

Theorem 3.

The joint functional $\Phi^{(2)}$ of the process $A(t)$ on the interval $[\tau_{\rho-1}, \tau_\rho)$ satisfies

$$\begin{aligned}
&\Phi^{(2)}(\theta, v_1, v_2, w, h_0, h) \\
&= \int_{t\geq 0} e^{-\theta t} E\left[e^{-v_1\cdot A_{\rho-1}-v_2\cdot A_\rho - w\cdot A(t) - h_0\tau_{\rho-1}-h\delta_\rho}\mathbf{1}_{\{\tau_{\rho-1}\leq t<\tau_{\rho-1}\}}\right] dt \\
&= D_y^{-1}\left(\frac{\gamma_0(v_1+v_2+y+w, \theta+h_0)}{1-\gamma(v_1+v_2+y+w, \theta+h_0)}\left[\frac{\gamma(v_2,h)-\gamma(v_2+w,h+\theta)}{\theta+\lambda(g(v_2+w)-g(v_2))}\right.\right. \\
&\qquad\left.\left. - \frac{\gamma(v_2+y,h)-\gamma(v_2+y+w,h+\theta)}{\theta+\lambda(g(v_2+y+w)-g(v_2+y))}\right]\right)(M_1, M_2)
\end{aligned}$$

where D_y^{-1} indicates the double inverse Fourier-Carson transform and γ represents the joint transform of the marks and inter-observation times.

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