Random Walks on Random Lattices
and Their Applications

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Dissertation Defense

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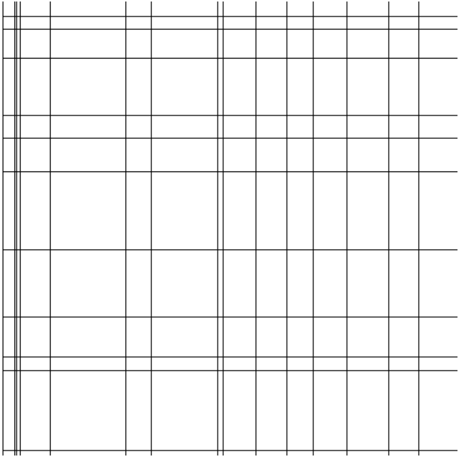
Random Walks on Random Lattices

(a) Random Walk



(b) Random Lattice

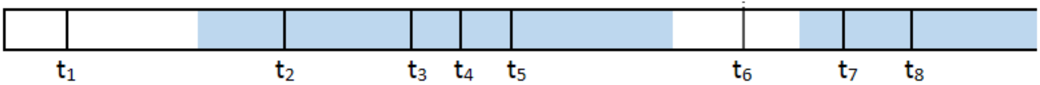
Stochastic Cumulative Loss Model

- Suppose (n_k, w_k) are *iid* random vectors valued in $\mathbb{N} \times \mathbb{R}_{\geq 0}$ with mutually dependent components
- Consider η, a Poisson random measure of rate λ: i.e. for a Poisson point process $0 < t_1 < t_2 < \ldots$ a.s. and $E \in \mathcal{B}(\mathbb{R}_{\geq 0})$,

$$\eta(E) = \sum_{k=1}^{\infty} (n_k, w_k) \varepsilon_{t_k}(E)$$

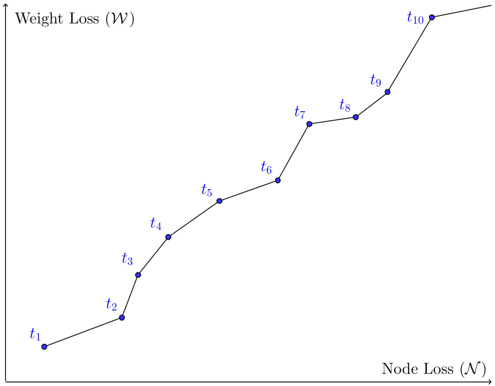
where ε_{t_k} is the Dirac (point mass) measure,

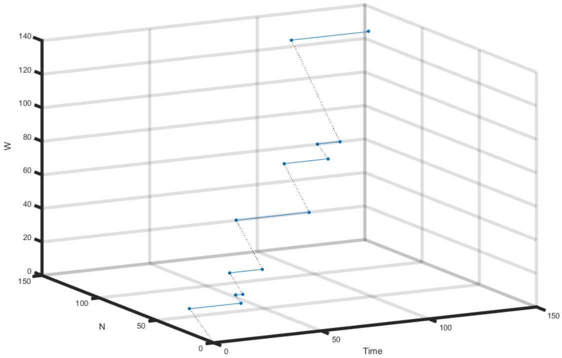
$$\varepsilon_{t_k}(E) = \begin{cases} 1, & \text{if } t_k \in E \\ 0, & \text{else} \end{cases}$$



Stochastic Cumulative Loss Process (cont.)

$\eta([0,t])$ is a monotone increasing process measuring the cumulative losses up to time t,





Delayed Observation

The process is observed upon a delayed renewal process

- $\mathcal{T} = \sum_{n=0}^{\infty} \varepsilon_{\tau_n}$
  - $\delta_k = \tau_k - \tau_{k-1}$ are *iid* with LST $l(\theta)$, $k \in \mathbb{N}$
  - $\tau_0 = \delta_0$ is independent of δ_k, $k \geq 1$

The process of interest is the **embedded process**

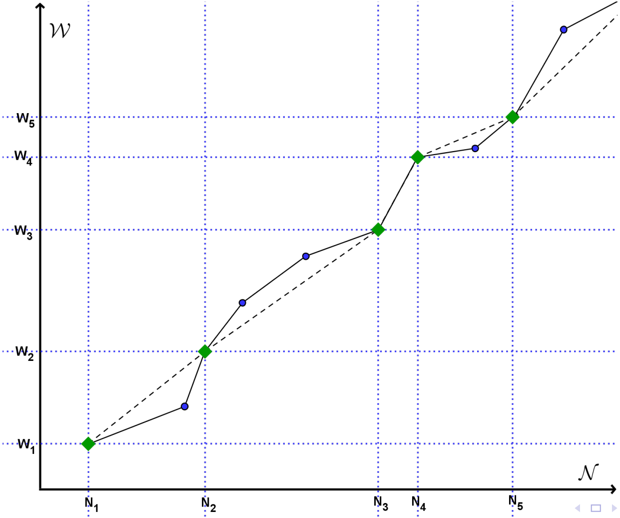
- $Z = \sum_{n=0}^{\infty} \eta((\tau_{n-1}, \tau_n])\varepsilon_{\tau_n} = \sum_{n=0}^{\infty} (X_n, Y_n)\varepsilon_{\tau_n}$
  - $(N_n, W_n) = Z([0, \tau_n])$, the value of the process at τ_n

Model Summary: 4 Random Parts

1. The arrival process $t_1 < t_2 < \dots$
2. The nodes lost per attack
3. The weight per node
4. The observation process $\tau_0 < \tau_1 < \tau_2 < \dots$

Delayed Observation as an Embedded Process

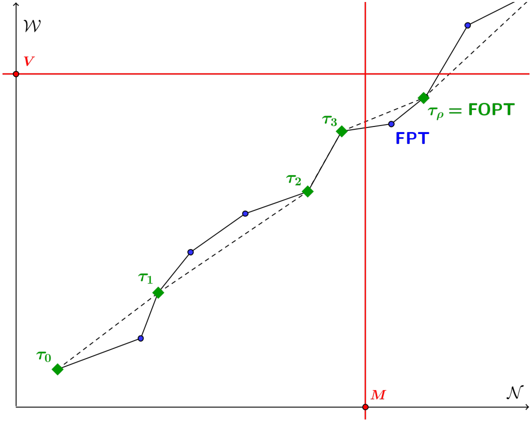
We consider the (green) embedded process moving on a random lattice



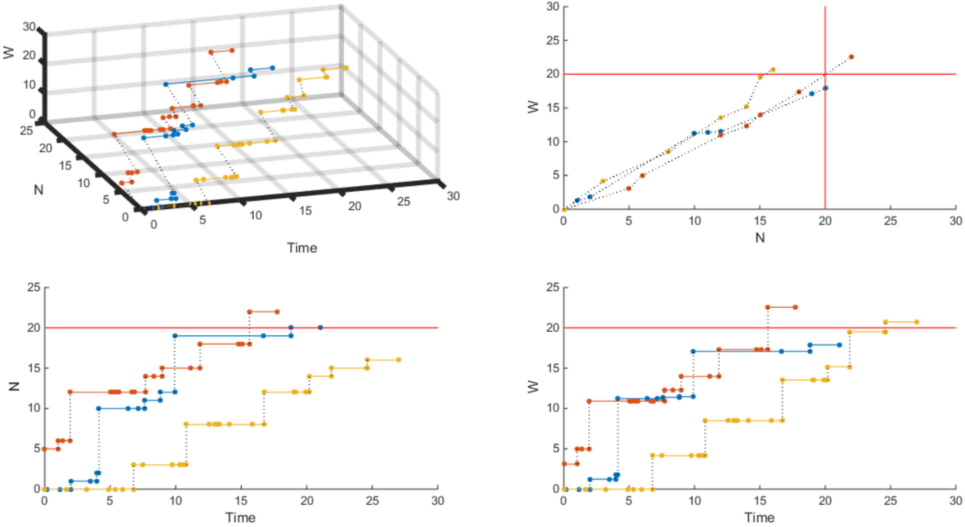
Observed Threshold Crossings

$$\rho = \min\{n : N_n > M \text{ or } W_n > V\}$$

- The **first observed passage time** (FOPT) is τ_ρ

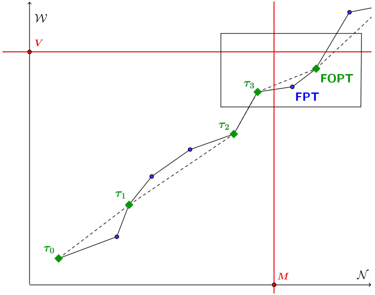


Simulations of the Observed Process



General Trajectory of the Upcoming Material

1. Analysis of the process upon $\tau_{\rho-1}$ and τ_ρ (time insensitive analysis)
   - J. H. Dshalalow and R. White. *Neural, Parallel, and Scientific Computations*, 21 (2013).
2. Strategy 1 for a refined analysis (auxiliary thresholds)
   - J. H. Dshalalow and R. White. *Stochastic Analysis and Applications*, 32:3 (2014).
3. d-dimensional time insensitive analysis (to be written as an additional paper)
4. Strategy 2 for refined analysis (time sensitive analysis)
   - J.H. Dshalalow and R. White. Time Sensitive Analysis of Independent and Stationary Increment Processes (2015 submitted paper).
   - R. White and J. H. Dshalalow. Random Walks on Random Lattices with Applications (2015 submitted paper).



Functional of Interest

- We seek a joint functional of the of the process upon $\tau_{\rho-1}$ and τ_ρ,

$$\Phi(\alpha_0, \alpha, \beta_0, \beta, h_0, h) = E\left[{\alpha_0}^{N_{\rho-1}} \alpha^{N_\rho} e^{-\beta_0 W_{\rho-1} - \beta W_\rho} e^{-h_0 \tau_{\rho-1} - h \tau_\rho}\right]$$

- Φ leads to marginal PGFs/LSTs,
  - $\Phi(1, \alpha, 0, 0, 0, 0) = E\left[\alpha^{N_\rho}\right]$
  - $\Phi(1, 1, \beta_0, 0, 0, 0) = E\left[e^{-\beta_0 W_{\rho-1}}\right]$
  - $\Phi(1, 1, 0, 0, 0, h) = E\left[e^{-h \tau_\rho}\right]$

  which lead to moments and distributions of components of the process upon $\tau_{\rho-1}$ and τ_ρ

Goal and Operational Calculus Strategy

- The goal is to derive Φ in an analytically or numerically tractable form
- Strategy to derive Φ,

$$\Phi \xrightarrow{\mathcal{H}} \Psi \xrightarrow{\text{Assumptions}} \Psi \text{ (convenient form)} \xrightarrow{\mathcal{H}^{-1}} \Phi \text{ (tractable)}$$

for an operator $\mathcal{H}$ introduced next

$\mathcal{H}_{pq}$ and $\mathcal{H}_{xy}^{-1}$ Operators

$$\mathcal{H}_{pq} = \mathcal{L}\mathcal{C}_q \circ D_p$$

$$\mathcal{L}\mathcal{C}_q(g(q))(y) = y\mathcal{L}_q(g(q)) = y \int_{q=0}^{\infty} g(q)e^{-qy}\,dq$$

$$D_p\{f(p)\}(x) = (1-x)\sum_{p=0}^{\infty} x^p f(p), \|x\| < 1$$

- D_p's inverse restores a sequence f,
  - $\mathcal{D}_x^k\left(\cdot\right) = \lim_{x\to 0} \frac{1}{k!}\frac{\partial^k}{\partial x^k}\left(\frac{1}{1-x}(\cdot)\right), k \in \mathbb{N}$
  - $\mathcal{D}_x^k\left(D_p\{f(p)\}(x)\right) = f(k)$

$$\mathcal{H}_{xy}^{-1}\left(\Psi(x,y)\right)(p,q) = \mathcal{L}_y^{-1}\left(\frac{1}{y}\mathcal{D}_x^p\left(\Psi(x,y)\right)\right)(q)$$

Decomposition of Φ

- Let $\mu = \inf\{n : N_n > M\}$, $\nu = \inf\{n : W_n > V\}$
- Decompose Φ as

$$\Phi = E\left[{\alpha_0}^{N_{\rho-1}} \alpha^{N_\rho} e^{-\beta_0 W_{\rho-1} - \beta W_\rho} e^{-h_0 \tau_{\rho-1} - h\tau_\rho}\right]$$

$$= E\left[\alpha_0^{N_{\mu-1}} \alpha^{N_\mu} e^{-\beta_0 W_{\mu-1} - \beta W_\mu - h_0 \tau_{\mu-1} - h\tau_\mu} \mathbf{1}_{\{\mu<\nu\}}\right]$$

$$+ E\left[\alpha_0^{N_{\mu-1}} \alpha^{N_\mu} e^{-\beta_0 W_{\mu-1} - \beta W_\mu - h_0 \tau_{\mu-1} - h\tau_\mu} \mathbf{1}_{\{\mu=\nu\}}\right]$$

$$+ E\left[\alpha_0^{N_{\nu-1}} \alpha^{N_\nu} e^{-\beta_0 W_{\nu-1} - \beta W_\nu - h_0 \tau_{\nu-1} - h\tau_\nu} \mathbf{1}_{\{\mu>\nu\}}\right]$$

$$= \Phi_{\mu<\nu} + \Phi_{\mu=\nu} + \Phi_{\mu>\nu}$$

- We will derive $\Phi_{\mu<\nu}$ of the **confined process** on the trace σ-algebra $\mathcal{F} \cap \{\mu < \nu\}$

Decomposing Further, Applying $\mathcal{H}_{pq}$

- Introduce $\mu(p) = \inf\{j : N_j > p\}$, $\nu(q) = \inf\{k : W_k > q\}$

$$\Phi_{\mu(p)<\nu(q)} = \sum_{j\geq 0} \sum_{k>j} E\left[\alpha_0^{N_{j-1}} \alpha^{N_j} e^{-\beta_0 W_{j-1} - \beta W_j - h_0 \tau_{j-1} - h \tau_j} \mathbf{1}_{\{\mu(p)=j,\, \nu(q)=k\}}\right]$$

Decomposing Further, Applying $\mathcal{H}_{pq}$

- Introduce $\mu(p) = \inf\{j : N_j > p\}$, $\nu(q) = \inf\{k : W_k > q\}$

$$\Phi_{\mu(p)<\nu(q)} = \sum_{j\geq 0}\sum_{k>j} E\left[\alpha_0^{N_{j-1}}\alpha^{N_j} e^{-\beta_0 W_{j-1}-\beta W_j - h_0\tau_{j-1}-h\tau_j}\mathbf{1}_{\{\mu(p)=j,\,\nu(q)=k\}}\right]$$

- By Fubini's Theorem, $\mathcal{H}_{pq}$ can be interchanged with the two series and expectation

- Using the memoryless property of the Poisson process and independent increments,

$$\begin{aligned}\mathcal{H}_{pq}\left(\Phi_{\mu(p)<\nu(q)}\right)(x,y) = & \sum_{j\geq 0} E\left[(\alpha_0\alpha x)^{N_{j-1}}\, e^{-(\beta_0+\beta+y)W_{j-1}-(h_0+h)\tau_{j-1}}\right] \\ & \times E\left[\alpha^{X_j}\left(1-x^{X_j}\right)e^{-(\beta+y)Y_j - h\delta_j}\right] \\ & \times \sum_{k>j} E\left[e^{-y\left(Y_{j+1}+\ldots+Y_{k-1}\right)}\right] E\left[1-e^{-yY_k}\right]\end{aligned}$$

Deriving $\Phi_{\mu<\nu}$

- Denote the joint transform of each observed increment

$$\gamma(z,v,\vartheta) = E\left[z^{X_1} e^{-vY_1 - \vartheta\delta_1}\right]$$

- The j and k series (assuming $\|\gamma\| < 1$) become

$$\mathit{\Gamma}_0^1 - \mathit{\Gamma}_0 + (\mathit{\Gamma}^1 - \mathit{\Gamma})\gamma_0 \sum_{j\geq 1} \gamma^{j-1} = \mathit{\Gamma}_0^1 - \mathit{\Gamma}_0 + \gamma_0 \frac{\mathit{\Gamma}^1 - \mathit{\Gamma}}{1-\gamma}$$

$$(1-\gamma(1,y,0)) \sum_{k>j} \gamma^{k-1-j}(1,y,0) = 1$$

- Then $\Phi_{\mu<\nu} = \mathcal{H}_{xy}^{-1}\left(\Gamma_0^1 - \Gamma_0 + \frac{\gamma_0}{1-\gamma}\left(\Gamma^1 - \Gamma\right)\right)(M,V)$, where

$$\gamma = \gamma\left(\alpha_0\alpha x, \beta_0 + \beta + y, h_0 + h\right)$$
$$\mathit{\Gamma} = \gamma\left(\alpha x, \beta + y, h\right)$$
$$\zeta^1 = \gamma(\alpha x, \beta, h)$$

Result for Φ

- Solving for $\Phi_{\mu=\nu}$ and $\Phi_{\mu>\nu}$ analogously and adding to $\Phi_{\mu<\nu}$,

$$\Phi = \mathcal{H}_{xy}^{-1}\left(\zeta_0^1 - \Gamma_0 + \frac{\gamma_0}{1-\gamma}\left(\zeta^1 - \Gamma\right)\right)(M, V)$$

where

$$\begin{aligned}\gamma &= \gamma\left(\alpha_0 \alpha x, \beta_0 + \beta + y, h_0 + h\right) \\ \Gamma &= \gamma\left(\alpha x, \beta + y, h\right) \\ \zeta^1 &= \gamma(\alpha x, \beta, h)\end{aligned}$$

Results for a Special Case

To demonstrate that tractable results can be derived from this, we will consider a special case where

- $\delta_k \in [\text{Exponential}(\mu)] \implies L(\theta) = \frac{\mu}{\mu+\theta}$
- $n_k \in [\text{Geometric}(a)] \implies g(z) = \frac{az}{1-bz}, (b = 1 - a)$
- $w_{jk} \in [\text{Exponential}(\xi)] \implies l(u) = \frac{\xi}{\xi+u}$
- $(N_0, W_0) = (0, 0)$

Theorem 1

$$\Phi\left(1,z,0,v,0,\theta\right)=E\left[z^{N_\rho}e^{-vW_\rho}e^{-\theta\tau_\rho}\right]$$

$$=1-\left(1-\frac{\mu}{\mu+\theta+\lambda}\frac{v+\xi(1-bz)}{v+\xi(1-c_2z)}\right)\left(1+\frac{b\mu}{\lambda+b\theta}+\frac{a\lambda\mu}{\left(\lambda+b\theta\right)\left(\lambda+\theta\right)}\phi\left(z,v,\theta\right)\right),$$

$$\phi(z,v,\theta)=\frac{v+\xi}{v+\xi(1-c_1z)}-\frac{c_1z\xi\boxed{Q(M-1,c_1z\xi V)}e^{-(v+\xi(1-c_1z))V}}{v+\xi(1-c_1z)}-\frac{(c_1z\xi)^M\boxed{P(M-1,(\xi+v)V)}}{(v+\xi(1-c_1z))(\xi+v)^{M-1}},$$

$$c_1=\frac{\lambda+b\theta}{\lambda+\theta},\qquad c_2=\frac{\lambda+b(\mu+\theta)}{\lambda+\mu+\theta},$$

and $Q(x,y)=\frac{\Gamma(x,y)}{\Gamma(x)}$ is the lower regularized gamma function.

Corollaries: Marginal Transforms

$$\Phi(1,z,0,0,0,0) = E\left[z^{N_\rho}\right] = \frac{zQ(M-1,z\xi V)e^{-\xi(1-z)V} + z^M P(M-1,\xi V)}{\mu+\lambda-(\lambda+b\mu)z}$$

$$\Phi(1,1,0,v,0,0) = E\left[e^{-vW_\rho}\right] = \frac{\lambda v + b\mu v + a\xi\mu\phi(1,v,0)}{a\xi\mu + (\lambda+\mu)v}$$

$$\Phi(1,1,0,0,0,\theta) = E\left[e^{-\theta\tau_\rho}\right] = 1 - \frac{\theta}{\mu+\theta}\left[1 + \frac{b\mu}{\lambda+b\theta} + \frac{a\lambda\mu\phi(1,0,\theta)}{(\lambda+b\theta)(\lambda+\theta)}\right]$$

Useful Results: CDF of First Observed Passage Time, τ_ρ

$$F_{\tau_\rho}(\vartheta) = P\left(\tau_\rho < \vartheta\right)$$
$$= \lambda P(M-1, \xi V) \sum_{j=0}^{M-1} c_j \phi_j(\vartheta) + \lambda e^{-\xi\lambda} \sum_{k=0}^{M-2} \frac{(\xi V)^k}{k!} \sum_{j=0}^{k} d_j \phi_j(\vartheta)$$

where

$$c_j = \binom{M-1}{j} (a\lambda)^j b^{M-1-j}$$
$$d_j = \binom{k}{j} (a\lambda)^j b^{k-j}$$
$$\phi_j(\vartheta) = \frac{1}{\lambda^{j+1}} P(j+1, \lambda\vartheta) - \frac{e^{-\mu\vartheta}}{(\lambda-\mu)^{j+1}} P(j+1, (\lambda-\mu)\vartheta)$$

Useful Results: Means at τ_ρ

$$E[N_\rho] = \frac{\lambda + b\mu}{a\mu} + M - (M-1)Q(M-1, \xi V) + \xi V Q(M-2, \xi V)$$

$$E[W_\rho] = \frac{E[N_\rho]}{\xi} = E[N_\rho]\, E[w_{11}]$$

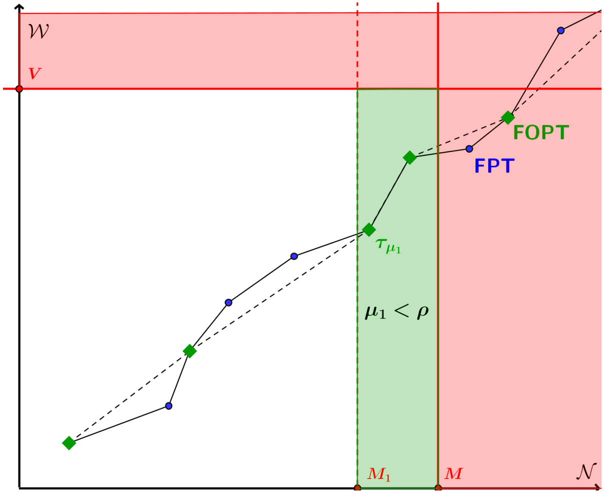
1,000 realizations of the process under each of the some sets of parameters $(\lambda, \mu, a, \xi, M, V)$ were simulated and sample means recorded:

Parameters	$E[N_\rho]$	S. Mean	A. Error	$E[W_\rho]$	S. Mean	A. Error
(1)	989.08	988.82	0.26	989.08	990.06	0.98
(2)	990.63	990.39	0.24	990.63	990.27	0.36
(3)	989.28	989.92	0.64	989.28	988.97	0.31
(4)	989.08	989.08	0.00	989.08	989.68	0.31
(5)	503.00	502.73	0.27	1006.00	1005.04	0.96
(6)	1002.00	1001.57	0.43	501.00	500.91	0.09
(7)	803.00	802.68	0.32	803.00	802.67	0.33
(8)	752.00	752.10	0.10	752.00	751.68	0.32
(9)	493.57	493.46	0.11	987.14	986.59	0.55

J. H. Dshalalow and R. White. *Neural, Parallel, and Scientific Computations*, 21 (2013).

Auxiliary Threshold Model

We introduce another threshold $M_1 < M$ with $\mu_1 = \min\{n : N_n > M_1\}$.



Functional of Interest

Our functional of interest will be similar to before, but containing terms associated with the auxiliary crossing at μ_1

$$\begin{aligned}\Phi_{\mu_1<\rho} &= \Phi_{\mu_1<\rho}\left(z_0, z, \alpha_0, \alpha, v_0, v, \beta_0, \beta, \theta_0, \theta, h_0, h\right) \\ &= E\left[\boxed{z_0^{N_{\mu_1-1}} z^{N_{\mu_1}}}\alpha_0^{N_{\rho-1}}\alpha^{N_\rho}\boxed{e^{-v_0 W_{\mu_1-1}-v W_{\mu_1}}}e^{-\beta_0 W_{\rho-1}-\beta W_\rho}\boxed{e^{-\theta_0\tau_{\mu_1-1}-\theta\tau_{\mu_1}}}e^{-h_0\tau_{\rho-1}-h\tau_\rho}\boxed{\mathbf{1}_{\{\mu_1<\rho\}}}\right] \\ &= \Phi_{\mu_1<\mu<\nu} + \Phi_{\mu_1<\mu=\nu} + \Phi_{\mu_1<\nu<\mu} \\ &= \left(\sum_{j\geq 0}\sum_{k>j}\sum_{n>k}\right) + \left(\sum_{j\geq 0}\sum_{n=k>j}\right) + \left(\sum_{j\geq 0}\sum_{n>j}\sum_{k>n}\right) \qquad (\mu_1 = j, \mu = k, \nu = n)\end{aligned}$$

We use an operator $\mathcal{H}_{pqs} = \mathcal{LC}_s \circ D_q \circ D_p$ adaped to work with the additional discrete threshold, but the path to results remains the same

$$\Phi_{\mu_1<\rho} \xrightarrow{\mathcal{H}_{pqs}} \Psi_{\mu_1<\rho} \xrightarrow{\text{Assumptions}} \Psi_{\mu_1<\rho}\ \text{(convenient)} \xrightarrow{\mathcal{H}^{-1}_{xyw}} \Phi_{\mu_1<\rho}\ \text{(tractable)}$$

Results for $\Phi_{\mu_1<\rho}$

$$\Phi_{\mu_1<\rho} = \mathcal{H}^{-1}_{xyw}\left(\left[\phi^1_0 - \phi_0 + \frac{\varphi_0}{1-\varphi}(\phi^1 - \phi)\right]\frac{\xi^1 - \chi}{1-\psi}\right)(M_1, M, V)$$

where

$$\varphi = \gamma\left(u_0 u \alpha_0 \alpha x y, v_0 + v + \beta_0 + \beta + w, \theta_0 + \theta + h_0 + h\right)$$
$$\phi = \gamma\left(u \alpha_0 \alpha x y, v + \beta_0 + \beta + w, \theta + h_0 + h\right)$$
$$\phi^1 = \gamma\left(u \alpha_0 \alpha y, v + \beta_0 + \beta + w, \theta + h_0 + h\right)$$
$$\psi = \gamma\left(\alpha_0 \alpha y, \beta_0 + \beta + w, h_0 + h\right)$$
$$\chi = \gamma\left(\alpha y, \beta + w, h\right)$$
$$\xi^1 = \gamma\left(\alpha, \beta, h\right)$$

Results for a Special Case

Suppose

- $\delta_k = c \; a.s.$
- n_k with **arbitrary** finite distribution $(p_1, p_2, ..., p_m)$
- $w_{jk} \in [\text{Gamma}(\alpha, \xi)]$

Theorem

$$\Phi_{\mu_1<\rho}(1,z,1,1,0,v,0,0,0,\theta,0,0) = \mathbb{E}\left[z^{N_{\mu_1}} e^{-vW_{\mu_1}} e^{-\theta\tau_{\mu_1}} \mathbf{1}_{\{\mu_1<\rho\}}\right]$$

$$= \left\{ \sum_{k=0}^{M_1-1} z^k F_k \sum_{m=0}^{M-1-k} z^m E_m \left(\frac{\xi}{v+\xi}\right)^{\alpha(k+m)} P(\alpha(k+m),(v+\xi)V) \right.$$

$$\left. - \sum_{k=0}^{M_1-1} z^k \left(\frac{\xi}{v+\xi}\right)^{\alpha k} P(\alpha k,(v+\xi)V) \sum_{n=0}^{k} E_n F_{k-n} \right\} e^{-c(\theta+\lambda)}$$

$$F_j = \sum_{r=0}^{\left\lfloor \frac{R-1}{R} j \right\rfloor} (c\lambda)^{j-r} \boxed{Li_{-(j-r)}\left(e^{-c(\theta+\lambda)}\right)} \sum_{\substack{\|\boldsymbol{\beta}\|_1=j \\ [R]\cdot\boldsymbol{\beta}=r+j}} \frac{p_1^{\beta_1}\cdots p_R^{\beta_R}}{\beta_1!\ \cdots\ \beta_R!},$$

$$E_j = \sum_{r=0}^{\left\lfloor \frac{R-1}{R} j \right\rfloor} (c\lambda)^{j-r} \sum_{\substack{\|\boldsymbol{\beta}\|_1=j \\ [R]\cdot\boldsymbol{\beta}=r+j}} \frac{p_1^{\beta_1}\cdots p_R^{\beta_R}}{\beta_1!\ \cdots\ \beta_R!}$$

A Computational Difficulty

$$\sum_{\substack{\|\boldsymbol{\beta}\|_1=j \\ [R]\cdot\boldsymbol{\beta}=r+j}} \frac{p_1^{\beta_1}\cdots p_R^{\beta_R}}{\beta_1!\,\cdots\,\beta_R!},$$

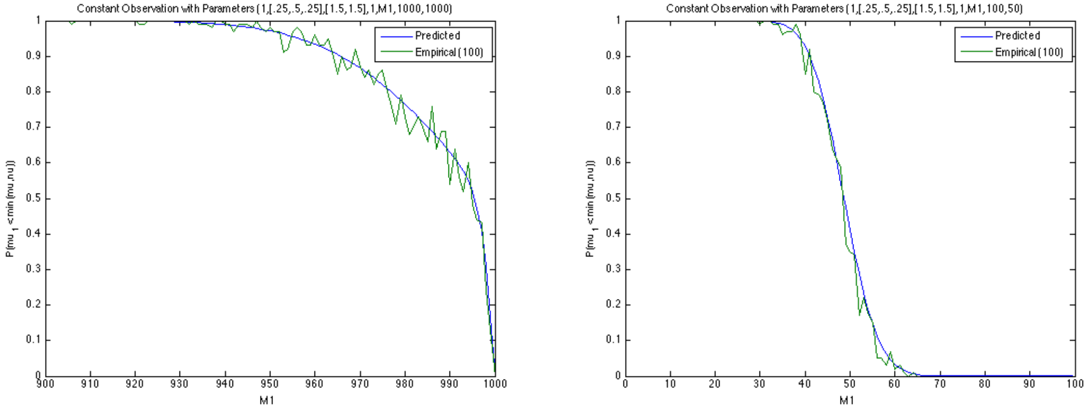
- Model 1 is general and useful, but it requires calculation of all integer solutions in $\{0, 1, ..., R\}^R$ of the linear system

$$\beta_1 + \beta_2 + ... + \beta_R = j$$
$$\beta_1 + 2\beta_2 + ... + R\beta_R = r + j$$

  for each $j = 0, 1, ..., M-1$ and $r = 0, 1, ..., \left\lfloor \frac{R-1}{R} j \right\rfloor$.
- Other special cases: n_k geometric, $n_k = n$ *a.s.*,

Results and Simulation

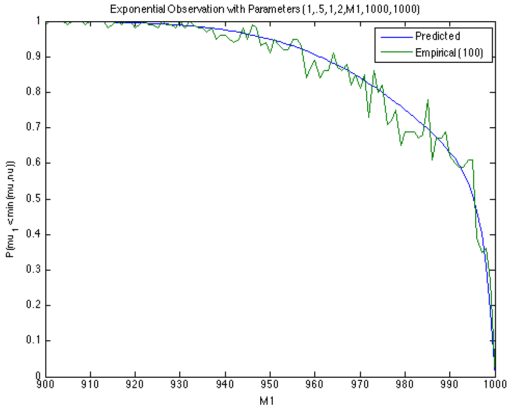
We find $\Phi_{\mu_1<\rho}(1,1,1,1,0,0,0,0,0,0,0,0) = \mathbb{E}\left[\mathbf{1}_{\{\mu_1<\rho\}}\right] = \mathbb{P}\{\mu_1 < \rho\}$ and compare to simulated results for a range of M_1 values

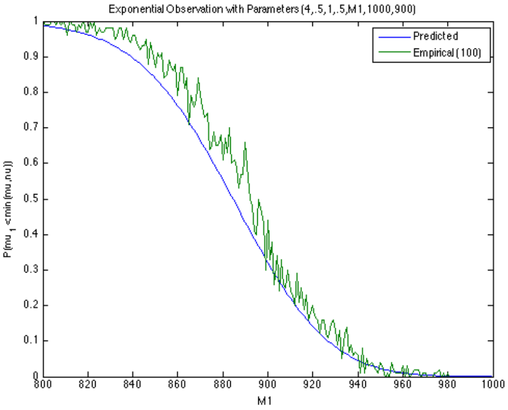


J. H. Dshalalow and R. White. *Stochastic Analysis and Applications*, 32:3 (2014).

Simulation for an Alternate Model

We did the same under the assumptions $\delta_1 \in [\text{Exponential}(\mu)]$, $n_1 \in [\text{Geometric}(a)]$, $w_{11} \in [\text{Exponential}(\xi)]$.





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Continuous and Dual Auxiliary Threshold Models

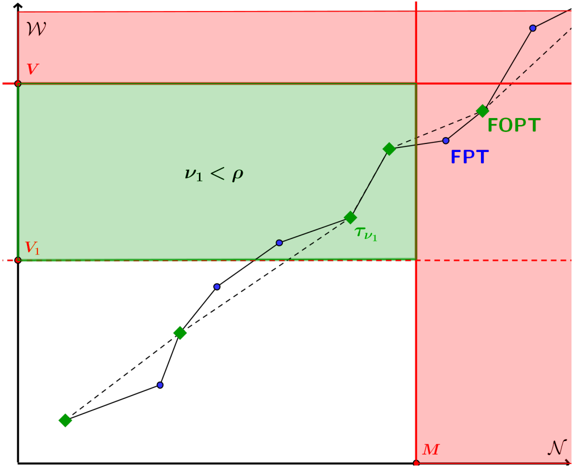


Figure: Continuous Auxiliary Model, $\nu_1 = \min\{n : W_n > V_1\}$

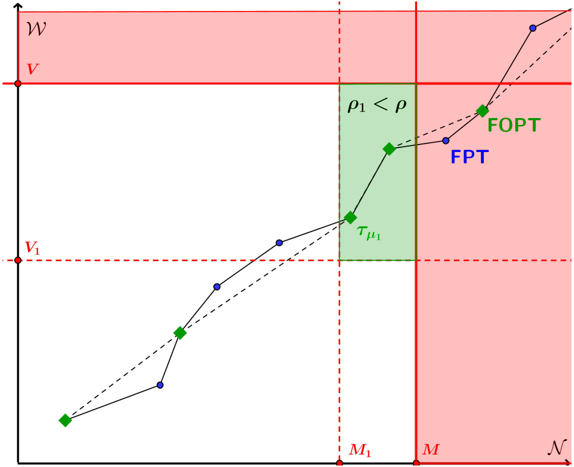


Figure: Dual Auxiliary Model, $\rho_1 = \max\{\mu_1, \nu_1\}$

d-dimensional Model

- Suppose $\mathbf{A}(t)$ is a d-dimensional process similar to the models above, with observed crossing indices $\{\nu_1, ..., \nu_d\}$ and $\rho = \min\{\nu_1, ..., \nu_d\}$. We investigate

$$\Phi(\mathbf{u}) = E\left[u_1^{\mathbf{A}_{1\rho}} \cdots u_d^{\mathbf{A}_{d\rho}}\right]$$

- If a dimension of the mark is continuous-valued, the operator D_p may be replaced with $\mathcal{L}\mathcal{C}_p$
- In addition, terms of the form $u_j^{\mathbf{A}_{j\rho}}$ in the functionals should be replaced with terms of the form $e^{-u_j \mathbf{A}_{j\rho}}$
  - i.e. take $u_j \mapsto e^{-u_j}$
- Thus, we lose no generality by assuming the process is discrete-valued

Confined Functionals

- Let $\mathcal{Q}$ be a sub-partial ordering of threshold crossings, then we seek

$$\Phi_{\mathcal{Q}}(\mathbf{u}) = E\left[u_1^{\mathbf{A}_{1\rho}} \cdots u_d^{\mathbf{A}_{d\rho}} \mathbf{1}_{\mathcal{Q}}\right]$$

- For example, let $\mathcal{Q} = \{\nu_1 = \nu_2 < \nu_3\} \cap \{\nu_4 < \nu_3\}$ in 4D, then we may find

$$\Phi_{\mathcal{Q}}(\mathbf{0}) = E\left[\mathbf{1}_{\mathcal{Q}}\right] = P\left(\nu_1 = \nu_2 < \nu_3, \nu_4 < \nu_3\right)$$

Confined Functionals

- For the set $\mathcal{P}$ partial orderings satisfying sub-partial ordering $\mathcal{Q}$,

$$\begin{aligned}\Phi_{\mathcal{Q}}(\mathbf{u}) &= E\left[\mathbf{u}^{\mathbf{A}_\rho}\mathbf{1}_{\mathcal{Q}}\right] \\ &= \sum_{\{P\in\mathcal{P}\}} E\left[\mathbf{u}^{\mathbf{A}_\rho}\mathbf{1}_{P}\right] \\ &= \sum_{\{P\in\mathcal{P}\}} \Phi_P(\mathbf{u})\end{aligned}$$

- For example, let $\mathcal{Q} = \{\nu_1 = \nu_2 < \nu_3\} \cap \{\nu_4 < \nu_3\}$ in 4D, then

$$\mathcal{P} = \{\{\nu_1 = \nu_2 < \nu_4 < \nu_3\}, \{\nu_1 = \nu_2 = \nu_4 < \nu_3\}, \{\nu_4 < \nu_1 = \nu_2 < \nu_3\}\}$$

$$\Phi_{\mathcal{P}} = \sum_{\{P\in\mathcal{P}\}} \Phi_P = \Phi_{\nu_1=\nu_2<\nu_4<\nu_3} + \Phi_{\nu_1=\nu_2=\nu_4<\nu_3} + \Phi_{\nu_4<\nu_1=\nu_2<\nu_3}$$

Arbitrary Confined Functional

- Suppose P is an *arbitrary* partial ordering of d crossings observed upon k times

$$\underbrace{\nu_{p(1)} = \ldots = \nu_{p(s_1)}}_{j_1} < \underbrace{\nu_{p(s_1+1)} = \ldots = \nu_{p(s_2)}}_{j_2} < \ldots < \underbrace{\nu_{p(s_{k-1}+1)} = \ldots = \nu_{p(d)}}_{j_k}$$

- Fixing j_n's, we derive

$$\Phi_P(\mathbf{u}) = \left(\underbrace{\sum_{j_k > j_{k-1}} \cdots \sum_{j_2 > j_1} \sum_{j_1 \geq 0}}_{k\ (\leq d)\ \text{series}} \Phi_{P_\mathbf{j}} \right) = \mathcal{H}_{\mathbf{y}}^{-1} \left(\left(R_{120} + \gamma_{p0} \frac{R_{121}}{1 - \gamma_p} \right) \prod_{j=2}^{k-1} \frac{R_{j2s_j}}{(1 - \gamma_{pj})} \right)$$

where R's simply depend on $\mathbf{u}$, $\mathbf{v}$, $\mathbf{y}$, and the increment's joint functional γ

R_{j2s_j}

$$R_{k2s_k} = E\left[\left(1 - y_{p\left(s_{k-1}+1\right)}^{\mathbf{A}_{s_k}^{p\left(s_{k-1}+1\right)}}\right)\cdots\left(1 - y_{p(d)}^{\mathbf{A}_{s_k}^{p(d)}}\right)\right]$$

$$= \Bigg\{1 - \left[\gamma_p\left(\mathbf{1}; y_{p\left(s_{k-1}+1\right)}, \mathbf{1}\right) + \gamma_p\left(\mathbf{1}; 1, y_{p\left(s_{k-1}+2\right)}, \mathbf{1}\right) + \ldots + \gamma_p\left(\mathbf{1}; \mathbf{1}, y_{p(n)}\right)\right]$$

$$+ \left[\gamma_p\left(\mathbf{1}; y_{p\left(s_{k-1}+1\right)}, y_{p\left(s_{k-1}+2\right)}, \mathbf{1}\right) + \ldots + \gamma_p\left(\mathbf{1}; y_{p\left(s_{k-1}+1\right)}, \mathbf{1}, y_{p(d)}\right) + \ldots + \gamma_p\left(\mathbf{1}; \mathbf{1}, y_{p(d-1)}, y_{p(d)}\right)\right]$$

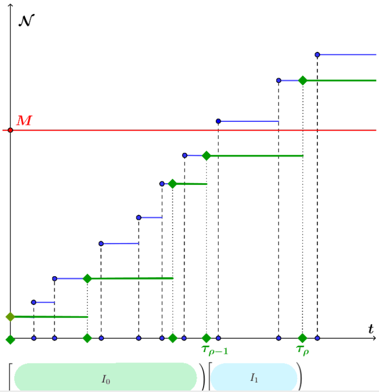
$$+ \ldots + (-1)^{r_k - r_{k-1}}\,\gamma_p\left(\mathbf{1}; y_{p\left(s_{k-1}+1\right)}, \ldots, y_{p(d)}\right)\Bigg\}$$

- Code (in Python) is developed that
  1. Generates all partial orderings of d indices ($d \leq 10$ feasible)
  2. Queries the list to prune it to $\mathcal{P}$ for any specified $\mathcal{Q}$
- The is computationally intense for large d, because we no only have $d!$ partial orderings with all $<$ signs, but also all partial orderings with $=$'s
- Special case results rely on d transform inversions, so this is sufficient for most purposes

Time Sensitive Analysis: Motivation

$$\Phi^{(1)} = \int_{t\geq 0} e^{-\theta t} E\left[e^{-\mathbf{v}_1\cdot\mathbf{A}_{\rho-1}-\mathbf{v}_2\cdot\mathbf{A}_\rho-\mathbf{w}\cdot\mathbf{A}(t)-h_0\tau_{\rho-1}-h\delta_\rho}\mathbf{1}_{\{t<\tau_{\rho-1}\}}\right] dt$$

$$\Phi^{(2)} = \int_{t\geq 0} e^{-\theta t} E\left[e^{-\mathbf{v}_1\cdot\mathbf{A}_{\rho-1}-\mathbf{v}_2\cdot\mathbf{A}_\rho-\mathbf{w}\cdot\mathbf{A}(t)-h_0\tau_{\rho-1}-h\delta_\rho}\mathbf{1}_{\{\tau_{\rho-1}\leq t<\tau_\rho\}}\right] dt$$



New capabilities:

- Joint results: $\mathbb{E}\left[e^{-uW_\rho}\mathbf{1}_{\{t<\tau_\rho\}}\right]$, $P\left(W_\rho < s, \tau_\rho < t\right)$, $P\left(N_\rho = n, \tau_\rho < t\right)$
- Conditional probabilities: $P\left(N_\rho = n | \tau_\rho > t\right) = \frac{P\left(N_\rho = n, \tau_\rho > t\right)}{P\left(\tau_\rho > t\right)}$

The Previous Method Fails!

- We would like to expand each functional in stochastic series as before

$$\mathcal{H}_{pq}\left(\int_{t\geq 0} e^{-\theta t} E\left[e^{-\mathbf{v}_1\cdot\mathbf{A}_{\rho-1}-\mathbf{v}_2\cdot\mathbf{A}_\rho-\mathbf{w}\cdot\mathbf{A}(t)-h_0\tau_{\rho-1}-h\delta_\rho}\mathbf{1}_{\{t<\tau_{\rho-1}\}}\right]dt\right)$$
$$=\mathcal{H}_{pq}\left(\sum_{j>0}\sum_{k>j}\int_{t\geq 0} e^{-\theta t} E\left[e^{-\mathbf{v}_1\cdot\mathbf{A}_{j-1}-\mathbf{v}_2\cdot\mathbf{A}_j-(\dots)}\mathbf{1}_{\{t<\tau_{j-1}\}}\mathbf{1}_{\{\mu(p)=j,\nu(q)=k\}}\right]dt\right)$$

- However, we cannot simply find the terms to sum since $\mathbf{A}(t)$ is involved
- Next, we prove a very general result leading to a way to represent these terms

ISI Processes

Consider a $\mathbb{R}^d$-valued stochastic process $\mathbf{A}(t)$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ such that

- $\mathbf{A}(t)$ has independent increments:
  - For $0 < t_1 < ... < t_k$, the increments $\mathbf{A}(t_1) - \mathbf{A}(0), \mathbf{A}(t_2) - \mathbf{A}(t_1), ..., \mathbf{A}(t_k) - \mathbf{A}(t_{k-1})$ are independent.
  - i.e., increments on non-overlapping time intervals are independent
- $\mathbf{A}(t)$ has stationary increments:
  - For $0 \leq s < t$, the distribution of $\mathbf{A}(t) - \mathbf{A}(s)$ depends only on $t - s$
- Lévy processes are ISI (e.g. Poisson processes, Wiener processes)

Discrete-valued Dimensions

- If a dimension of the mark is discrete-valued, the operator $\mathcal{LC}_p$ may be replaced with the D_p operator
- In addition, terms of the form $e^{-v \cdot \mathbf{A}_n}$ in the functionals should be replaced with terms of the form $\prod_{j=0}^{d} \mathbf{v}_j^{A_{jn}}$ (i.e. take $\mathbf{v}_j \mapsto -\ln \mathbf{v}_j$).
- Thus, we lose no generality by assuming the process is continuous-valued in each dimension.

ISI Functionals

- Suppose $\mathcal{T} = \{T_0, T_1, ..., T_m\}$ are random variables with $T_n = T_{n-1} + \Delta_n$ where each Δ_n is independent of $\Delta_0, \Delta_1, ..., \Delta_{n-1}$
  - We do **not** assume they are identically distributed
- For $1 \leq n \leq m$, we will seek functionals of the form

$$F_n\left(t, \mathbf{v}_0, ..., \mathbf{v}_m, \mathbf{w}, \mathbf{x}\right) = E\left[e^{-\sum_{j=0}^{m} \mathbf{v}_j \cdot \mathbf{A}\left(T_j\right) - \mathbf{w} \cdot \mathbf{A}(t) - \mathbf{x} \cdot \boldsymbol{\Delta}} \mathbf{1}_{\left\{T_{n-1} \leq t < T_n\right\}}\right],$$

where $\boldsymbol{\Delta} = (\Delta_0, ..., \Delta_m)$

Theorem 1

The functional $F_n(t, \mathbf{v}_0, ..., \mathbf{v}_m, \mathbf{w}, \mathbf{x})$ of the d-dimensional ISI process $\mathbf{A}(t)$ on the trace σ-algebra $\mathcal{F} \cap \{T_{n-1} \leq t < T_n\}$ where T_{n-1} and Δ_n are independent of $\mathcal{F}_t$ satisfies

$$F_n^*(\theta, \mathbf{v}_0, ..., \mathbf{v}_m, \mathbf{w}, \mathbf{x}) = \prod_{j=0}^{n-1} \gamma_j(\mathbf{b}_j + \mathbf{w}, x_j + \theta) E\left[e^{-x_n \Delta_n} \psi(\mathbf{b}_n, \mathbf{b}_n + \mathbf{w}, \Delta_n)\right] \prod_{j=n+1}^{m} \gamma_j(\mathbf{b}_j, x_j)$$

where

$$\mathbf{b}_j = \sum_{i=j}^{m} \mathbf{v}_i$$

$$\varphi(\mathbf{b}, s) = E\left[e^{-\mathbf{b}\cdot\mathbf{A}(s)}\right]$$

$$\psi(\mathbf{b}, \mathbf{x}, \alpha) = \left(e^{-\theta(\cdot)}\varphi(\mathbf{b}, \cdot)\right) * \varphi(\mathbf{x}, \cdot)(\alpha) = \int_0^\alpha e^{-\theta t}\varphi(\mathbf{b}, t)\varphi(\mathbf{x}, \alpha - t)\, dt$$

$$\gamma_j(\mathbf{a}, \vartheta) = E\left[e^{-\vartheta\Delta_j} e^{-\mathbf{a}\cdot\left[\mathbf{A}(T_j) - \mathbf{A}(T_{j-1})\right]}\right] = E\left[e^{-\mathbf{a}\cdot\mathbf{A}(\Delta_j) - \vartheta\Delta_j}\right]$$

Theorem 1 Proof

$$\mathbf{A}\left(T_k\right) = \sum_{j=0}^{k} \left(\mathbf{A}\left(T_j\right) - \mathbf{A}\left(T_{j-1}\right)\right), \qquad k = 0, ..., n-1$$

$$\mathbf{A}(t) = \mathbf{A}(t) - \mathbf{A}\left(T_{n-1}\right) + \sum_{j=0}^{n-1} \left(\mathbf{A}\left(T_j\right) - \mathbf{A}\left(T_{j-1}\right)\right)$$

$$\mathbf{A}\left(T_k\right) = \sum_{j=0}^{k} \left(\mathbf{A}\left(T_j\right) - \mathbf{A}\left(T_{j-1}\right)\right) + \mathbf{A}(t) - \mathbf{A}(t), \qquad k = n, ..., m$$

Let $\boldsymbol{\alpha} = (\alpha_0, ..., \alpha_m)$ and $s_n = \sum_{j=0}^{n} \alpha_j$. Writing the expectation F_n^* explicitly,

$$\begin{aligned} &F_n^*\left(\theta, \mathbf{v}_0, ..., \mathbf{v}_m, \mathbf{w}, \mathbf{x}\right) \\ &= \int_{t\geq 0} e^{-\theta t} \int_{\boldsymbol{\alpha}\in\mathbb{R}_{\geq 0}^{m+1}} e^{-\mathbf{x}\cdot\boldsymbol{\alpha}} E\Bigg[e^{-\sum_{j=0}^{n-1}\left(\mathbf{b}_j+\mathbf{w}\right)\cdot\left[\mathbf{A}\left(s_j\right)-\mathbf{A}\left(s_{j-1}\right)\right]-\left(\mathbf{b}_n+\mathbf{w}\right)\cdot\left[\mathbf{A}(t)-\mathbf{A}\left(s_{n-1}\right)\right]} \\ &\qquad \times e^{-\mathbf{b}_n\cdot\left[\mathbf{A}\left(s_n\right)-\mathbf{A}(t)\right]-\sum_{j=n+1}^{m}\mathbf{b}_j\cdot\left[\mathbf{A}\left(s_j\right)-\mathbf{A}\left(s_{j-1}\right)\right]} \\ &\qquad \times \mathbf{1}_{\left\{s_{n-1}\leq t< s_{n-1}+\alpha_n\right\}}\Bigg] \, dP_{\bigotimes_{j=0}^{m}\Delta_j}\left(\alpha_0, ..., \alpha_m\right) dt \end{aligned}$$

Theorem 1 Proof (cont.)

By the independent and stationary increment properties,

$$F_n^*\left(\theta, \mathbf{v}_0, ..., \mathbf{v}_m, \mathbf{w}, \mathbf{x}\right)$$

$$= \int\limits_{t \geq 0} e^{-\theta t} \int\limits_{\boldsymbol{\alpha} \in \mathbb{R}_{\geq 0}^{m+1}} e^{-\mathbf{x} \cdot \boldsymbol{\alpha}} \prod_{j=0}^{n-1} \varphi\left(\mathbf{b}_j + \mathbf{w}, \alpha_j\right) \varphi\left(\mathbf{b}_n + \mathbf{w}, t - s_{n-1}\right) \varphi\left(\mathbf{b}_n, s_{n-1} + \alpha_n - t\right)$$

$$\times \prod_{j=n+1}^{m} \varphi\left(\mathbf{b}_j, \alpha_j\right) \mathbf{1}_{\left\{s_{n-1} \leq t < s_{n-1} + \alpha_n\right\}} \, dP_{\bigotimes_{j=0}^{m} \Delta_j}(\alpha_0, ..., \alpha_m) \, dt$$

Theorem 1 Proof (cont.)

By the independent and stationary increment properties,

$$F_n^* (\theta, \mathbf{v}_0, ..., \mathbf{v}_m, \mathbf{w}, \mathbf{x})$$

$$= \int\limits_{t \geq 0} e^{-\theta t} \int\limits_{\boldsymbol{\alpha} \in \mathbb{R}_{\geq 0}^{m+1}} e^{-\mathbf{x} \cdot \boldsymbol{\alpha}} \prod_{j=0}^{n-1} \varphi (\mathbf{b}_j + \mathbf{w}, \alpha_j) \varphi (\mathbf{b}_n + \mathbf{w}, t - s_{n-1}) \varphi (\mathbf{b}_n, s_{n-1} + \alpha_n - t)$$

$$\times \prod_{j=n+1}^{m} \varphi (\mathbf{b}_j, \alpha_j) \mathbf{1}_{\{s_{n-1} \leq t < s_{n-1} + \alpha_n\}} \, dP_{\bigotimes_{j=0}^{m} \Delta_j} (\alpha_0, ..., \alpha_m) \, dt$$

By Fubini's Theorem and the independence of $\Delta_0, ..., \Delta_m$,

$$= \prod_{j=0}^{n-1} \int\limits_{\alpha_j \geq 0} e^{-(x_j + \theta) \alpha_j} \varphi (\mathbf{b}_j + \mathbf{w}, \alpha_j) \, dP_{\Delta_j} (\alpha_j) \prod_{j=n+1}^{m} \int\limits_{\alpha_j \geq 0} e^{-x_j \alpha_j} \varphi (\mathbf{b}_j, \alpha_j) \, dP_{\Delta_j} (\alpha_j)$$

$$\times \int_{\alpha_n \geq 0} e^{-x_n \alpha_n} \int_{t = s_{n-1}}^{s_{n-1} + \alpha_n} e^{-\theta (t - s_{n-1})} \varphi (\mathbf{b}_n + \mathbf{w}, t - s_{n-1}) \varphi (\mathbf{b}_n, s_{n-1} + \alpha_n - t) \, dt \, dP_{\Delta_n} (\alpha_n)$$

Theorem 1 Proof (cont.)

By the translation invariance of the Lebesgue measure, taking $u = t - s_{n-1}$,

$$F_n^* \left(\theta, \mathbf{v}_0, ..., \mathbf{v}_m, \mathbf{w}, \mathbf{x}\right)$$

$$= \prod_{j=0}^{n-1} \gamma_j \left(\mathbf{b}_j + \mathbf{w}, x_j + \theta\right) \prod_{j=n+1}^{m} \gamma_j \left(\mathbf{b}_j, x_j\right)$$

$$\times \int_{\alpha_n \geq 0} e^{-x_n \alpha_n} \boxed{\int_{u=0}^{\alpha_n} e^{-\theta u} \varphi \left(\mathbf{b}_n + \mathbf{w}, u\right) \varphi \left(\mathbf{b}_n, \alpha_n - u\right) du} dP_{\Delta_n} \left(\alpha_n\right) \quad (1)$$

$$= \prod_{j=0}^{n-1} \gamma_j \left(\mathbf{b}_j + \mathbf{w}, x_j + \theta\right) \prod_{j=n+1}^{m} \gamma_j \left(\mathbf{b}_j, x_j\right) \int_{\alpha_n \geq 0} e^{-x_n \alpha_n} \psi \left(\mathbf{b}_n + \mathbf{w}, \mathbf{b}_n, \alpha_n\right) dP_{\Delta_n} \left(\alpha_n\right) \quad (2)$$

$$= \prod_{j=0}^{n-1} \gamma_j \left(\mathbf{b}_j + \mathbf{w}, x_j + \theta\right) E \left[e^{-x_n \Delta_n} \psi \left(\mathbf{b}_n + \mathbf{w}, \mathbf{b}_n, \Delta_n\right)\right] \prod_{j=n+1}^{m} \gamma_j \left(\mathbf{b}_j, x_j\right) \quad (3)$$

Corollary 2

Let $\mathbf{A}(t)$ be a d-dimensional marked Poisson process of rate λ and assume $\mathcal{T}$ is independent of $\mathcal{F}_t$, then

$$F_n^*\left(\theta, \mathbf{v}_0, \ldots, \mathbf{v}_m, \mathbf{w}, \mathbf{x}\right) = \prod_{j=0}^{n-1} L_j\left(\theta + x_j + \lambda\left(1 - g\left(\mathbf{b}_j + \mathbf{w}\right)\right)\right)$$
$$\times \frac{L_n\left(x_n + \lambda\left(1 - g\left(\mathbf{b}_n\right)\right)\right) - L_n\left(x_n + \theta + \lambda\left(1 - g\left(\mathbf{b}_n + \mathbf{w}\right)\right)\right)}{\theta + \lambda\left(g\left(\mathbf{b}_n + \mathbf{w}\right) - g\left(\mathbf{b}_n\right)\right)}$$
$$\times \prod_{j=n+1}^{m} L_j\left(x_j + \lambda\left(1 - g\left(\mathbf{b}_j\right)\right)\right)$$

where

- $L_j\left(z\right) = E\left[e^{-z\Delta_j}\right]$ is the LST of Δ_j
- $g(\mathbf{v}) = E\left[e^{-\mathbf{v}\cdot\mathbf{a}_1}\right]$ is a joint LST of the marks
    - i.e. assume the marks are nonnegative real-valued

Returning to the Goal

- The goal is to find joint functionals of the form

$$\Phi^{(1)}(\theta, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}, h_0, h) = \int_{t \geq 0} e^{-\theta t} E\left[e^{-\mathbf{v}_1 \cdot \mathbf{A}_{\rho-1} - \mathbf{v}_2 \cdot \mathbf{A}_{\rho} - \mathbf{w} \cdot \mathbf{A}(t) - h_0 \tau_{\rho-1} - h \delta_\rho} \mathbf{1}_{\{t < \tau_{\rho-1}\}}\right] dt$$

$$\Phi^{(2)}(\theta, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}, h_0, h) = \int_{t \geq 0} e^{-\theta t} E\left[e^{-\mathbf{v}_1 \cdot \mathbf{A}_{\rho-1} - \mathbf{v}_2 \cdot \mathbf{A}_{\rho} - \mathbf{w} \cdot \mathbf{A}(t) - h_0 \tau_{\rho-1} - h \delta_\rho} \mathbf{1}_{\{\tau_{\rho-1} \leq t < \tau_\rho\}}\right] dt$$

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$$\Phi^{(2)}(\theta, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}, h_0, h) = \int_{t\geq 0} e^{-\theta t} E\left[e^{-\mathbf{v}_1\cdot\mathbf{A}_{\rho-1}-\mathbf{v}_2\cdot\mathbf{A}_{\rho}-\mathbf{w}\cdot\mathbf{A}(t)-h_0\tau_{\rho-1}-h\delta_\rho}\mathbf{1}_{\{\tau_{\rho-1}\leq t<\tau_\rho\}}\right] dt$$

- $\tau_{\rho-1}$ and δ_ρ are not independent, so the results are not immediately applicable. Let $T_0 = 0$ and

$$T_1 = \tau_{j-1} = \delta_0 + \delta_1 + ... + \delta_{j-1} \qquad T_2 = \tau_j = T_1 + \delta_j$$

$$T_3 = \tau_{k-1} = T_2 + \delta_{j+1} + ... + \delta_{k-1} \qquad T_4 = \tau_k = T_3 + \delta_k$$

for fixed j and k so that each Δ_n is independent of the prior Δ_r's

Corollary 3

Let $\mathbf{A}(t)$ be a 2-dimensional marked Poisson process of rate λ. For the process with $m = 4$ on the trace σ-algebra $\mathcal{F}(\Omega) \cap \{t < \tau_{j-1}\}$, respectively,

$$F^*_{1jk}(\theta, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{w}, h_0, h_0, h)$$

$$= \mathcal{L}_t\left\{E\left[e^{-\mathbf{v}_1\cdot\mathbf{A}_{j-1}-\mathbf{v}_2\cdot\mathbf{A}_j-\mathbf{v}_3\cdot\mathbf{A}_{k-1}-\mathbf{v}_4\cdot\mathbf{A}_k-\mathbf{w}\cdot\mathbf{A}(t)-h_0\tau_{j-1}-h\delta_j}\mathbf{1}_{\left\{t<\tau_{j-1}\right\}}\mathbf{1}_{\{\mu=j,\nu=k\}}\right]\right\}$$

$$= \frac{\gamma_0(\mathbf{b}_1, h_0)\gamma^{j-1}(\mathbf{b}_1, h_0) - \gamma_0(\mathbf{b}_1+\mathbf{w}, \theta+h_0)\gamma^{j-1}(\mathbf{b}_1+\mathbf{w}, \theta+h_0)}{\theta+\lambda(g(\mathbf{b}_1+\mathbf{w}) - g(\mathbf{b}_1))}\gamma(\mathbf{b}_2, h)\gamma(\mathbf{v}_4, 0)\gamma^{k-1-j}(\mathbf{b}_3, 0)$$

where

$$\mathbf{A}_n = \mathbf{A}(\tau_n)$$

$$\mathbf{X}_n = \mathbf{A}_n - \mathbf{A}_{n-1}$$

$$\gamma(\mathbf{v}, \theta) = E\left[e^{-\mathbf{v}\cdot\mathbf{X}_1-\theta\delta_1}\right]$$

Theorem 4

The joint functional $\Phi_{\mu<\nu}^{(1)}$ of the process $\mathbf{A}(t)$ on the interval $[0, \tau_{\rho-1})$ satisfies

$$\Phi_{\mu<\nu}^{(1)}(\theta, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}, h_0, h)$$

$$= \int_{t\geq 0} e^{-\theta t} E\left[e^{-\mathbf{v}_1\cdot\mathbf{A}_{\mu-1}-\mathbf{v}_2\cdot\mathbf{A}_{\mu}-\mathbf{w}\cdot\mathbf{A}(t)-h_0\tau_{\mu-1}-h\delta_{\mu}}\mathbf{1}_{\left\{t<\tau_{\mu-1}\right\}}\mathbf{1}_{\{\mu<\nu\}}\right] dt$$

$$= \mathcal{H}_{y_1 y_2}^{-1}\left(\frac{\gamma\left(\left(v_{21}, v_{22}+y_2\right), h\right)-\gamma\left(\mathbf{v}_2+\mathbf{y}, h\right)}{\theta+\lambda\left(g\left(\mathbf{v}_1+\mathbf{v}_2+\mathbf{y}+\mathbf{w}\right)-g\left(\mathbf{v}_1+\mathbf{v}_2+\mathbf{y}\right)\right)}\right.$$

$$\left.\times\left[\frac{\gamma_0\left(\mathbf{v}_1+\mathbf{v}_2+\mathbf{y}, h_0\right)}{1-\gamma\left(\mathbf{v}_1+\mathbf{v}_2+\mathbf{y}, h_0\right)}-\frac{\gamma_0\left(\mathbf{v}_1+\mathbf{v}_2+\mathbf{y}+\mathbf{w}, \theta+h_0\right)}{1-\gamma\left(\mathbf{v}_1+\mathbf{v}_2+\mathbf{y}+\mathbf{w}, \theta+h_0\right)}\right]\right)(M, V)$$

where

$$\mathcal{H}_{pq} = \mathcal{LC}_p \circ \mathcal{LC}_q$$

Theorem 4 Proof

$$\mathcal{H}_{pq}\left(\Phi^{(1)}_{\mu(p)<\nu(q)}(\theta, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}, h_0, h)\right)$$
$$= \mathcal{H}_{pq}\left(\sum_{j>0}\sum_{k>j}\int_{t\geq 0} e^{-\theta t} E\left[e^{-\mathbf{v}_1\cdot\mathbf{A}_{j-1}-\mathbf{v}_2\cdot\mathbf{A}_j-(\ldots)}\mathbf{1}_{\{t<\tau_{j-1}\}}\mathbf{1}_{\{\mu(p)=j,\nu(q)=k\}}\right]dt\right)$$

Theorem 4 Proof

$$\mathcal{H}_{pq}\left(\Phi^{(1)}_{\mu(p)<\nu(q)}\left(\theta,\mathbf{v}_1,\mathbf{v}_2,\mathbf{w},h_0,h\right)\right)$$
$$=\mathcal{H}_{pq}\left(\sum_{j>0}\sum_{k>j}\int_{t\geq 0}e^{-\theta t}E\left[e^{-\mathbf{v}_1\cdot\mathbf{A}_{j-1}-\mathbf{v}_2\cdot\mathbf{A}_j-(\dots)}\mathbf{1}_{\left\{t<\tau_{j-1}\right\}}\mathbf{1}_{\{\mu(p)=j,\nu(q)=k\}}\right]dt\right)$$

Then, by Fubini's Theorem and since $1_{\left\{t<\tau_{j-1}\right\}}=0$ if $j=0$,

$$=\sum_{j>0}\sum_{k>j}F^*_{1jk}\left(\theta,\mathbf{v}_1+\left(y_1,0\right),\mathbf{v}_2,\left(0,y_2\right),\mathbf{0},\mathbf{w},h_0,h_0,h\right)$$
$$-\sum_{j>0}\sum_{k>j}F^*_{1jk}\left(\theta,\mathbf{v}_1,\mathbf{v}_2+\left(y_1,0\right),\left(0,y_2\right),\mathbf{0},\mathbf{w},h_0,h_0,h\right)$$
$$-\sum_{j>0}\sum_{k>j}F^*_{1jk}\left(\theta,\mathbf{v}_1+\left(y_1,0\right),\mathbf{v}_2,\mathbf{0},\left(0,y_2\right),\mathbf{w},h_0,h_0,h\right)$$
$$+\sum_{j>0}\sum_{k>j}F^*_{1jk}\left(\theta,\mathbf{v}_1,\mathbf{v}_2+\left(y_1,0\right),\mathbf{0},\left(0,y_2\right),\mathbf{w},h_0,h_0,h\right)$$

which by Corollary 3 converge to the proper terms assuming $\|\gamma\|<1$

5 Similar Functionals

- Analogous proofs yield results leading to

$$\Phi^{(1)} = \Phi^{(1)}_{\mu<\nu} + \Phi^{(1)}_{\mu=\nu} + \Phi^{(1)}_{\mu>\nu}$$

- After deriving another corollary to Theorem 2 analogous to Corollary 3 on $\{\tau_{\rho-1} \leq t < \tau_\rho\}$, we also find similar expressions for

$$\Phi^{(2)} = \Phi^{(2)}_{\mu<\nu} + \Phi^{(2)}_{\mu=\nu} + \Phi^{(2)}_{\mu>\nu}$$

Deriving a Joint Probability Distribution (1-d Model)

1. Derive a corollary to Theorem 1 with $d = 1$, $m = 2$ (as in Corollary 3)
2. Expand in series under $\mathcal{H}_p$ (as in Theorem 4)
3. Specify g and l and take the inverse $\mathcal{D}_y^{M-1}$ to get

$$\Phi^{(1)*}(\theta, 1, u, 1, 0, 0) = \mathcal{L}_t \left\{ \mathbb{E}\left[u^{A_\rho} \mathbf{1}_{\{t<\tau_{\rho-1}\}} \right] \right\} (\theta)$$

4. Find the inverse Laplace transform to find

$$\Phi^{(1)}(\theta, 1, u, 1, 0, 0) = \mathbb{E}\left[u^{A_\rho} \mathbf{1}_{\{t<\tau_{\rho-1}\}} \right]$$

5. This a restricted PGF, so apply $\frac{1}{r!} \lim\limits_{u\to 0} \frac{\partial^r}{\partial u^r}(\cdot)$ to find

$$\mathbb{P}\{N_\rho = r, \tau_{\rho-1} > t\}$$

Joint Distribution of $\tau_{\rho-1}$ and A_ρ (1-d Model)

$$
\begin{aligned}
&P\left\{A_\rho = r, \tau_{\rho-1} > t\right\} \\
&= \frac{\mu}{\lambda}\frac{a\mu}{\mu+\lambda}\left[\frac{\mu+\lambda}{a\mu}R_{0r} + \sum_{j=1}^{M-1} R_{jr}\right] - \frac{\mu}{\mu+\lambda}\left[G_0 R_{0r} + \sum_{j=1}^{M-1}\left(G_j - H_{j-1}\right)R_{jr}\right] \\
&\quad - \frac{\mu}{\lambda}\left[\sum_{j=0}^{M-1}\sum_{i=0}^{M-1-j} c^i \mathbf{1}_{\{r=i+j\}} - (b+c)\sum_{j=0}^{M-2}\sum_{i=0}^{M-2-j} c^i \mathbf{1}_{\{r=i+j+1\}} + bc\sum_{j=0}^{M-3}\sum_{i=0}^{M-3-j} c^i \mathbf{1}_{\{r=i+j+2\}}\right] \\
&\quad + \frac{\mu}{\mu+\lambda}\left[\sum_{j=0}^{M-1} G_j \sum_{i=0}^{M-1-j} c^i \mathbf{1}_{\{r=i+j\}} - \sum_{j=0}^{M-2}\left(bG_j + H_j\right)\sum_{i=0}^{M-2-j} c^i \mathbf{1}_{\{r=i+j+1\}} + b\sum_{j=0}^{M-3} H_j \sum_{i=0}^{M-3-j} c^i \mathbf{1}_{\{r=i+j+2\}}\right]
\end{aligned}
$$

where

$$c = F(\mu, 1) = \frac{b\mu+\lambda}{\mu+\lambda}$$

$$R_{jr} = \begin{cases} 0, & \text{if } r < j \\ 1, & \text{if } r = j \\ (c-b)\,c^{r-j-1}, & \text{if } r > j \end{cases}$$

An Interesting Generalization

- Suppose during the the random time interval $I_j = (T_{j-1}, T_j)$, then

$$g_j(\mathbf{v}) = E\left[e^{-\mathbf{v}\cdot\mathbf{a}_{j1}}\right]$$

$$l_j(\vartheta) = E\left[e^{-\vartheta\delta_{j1}}\right]$$

$$\gamma_j(\mathbf{v}, \vartheta) = E\left[e^{-\mathbf{v}\cdot\mathbf{X}_{j1}-\vartheta\delta_{j1}}\right]$$

- Then it is simple to rework results 2-4 above
- This allows the position independence of the marks to be relaxed